

CONFORMAL SYMMETRY BREAKING DIFFERENTIAL OPERATORS ON DIFFERENTIAL FORMS

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ABSTRACT. We study conformal symmetry breaking differential operators which map differential forms on \mathbb{R}^n to differential forms on a codimension one subspace \mathbb{R}^{n-1} . These operators are equivariant with respect to the conformal Lie algebra of the subspace \mathbb{R}^{n-1} . They correspond to homomorphisms of generalized Verma modules for $\mathfrak{so}(n, 1)$ into generalized Verma modules for $\mathfrak{so}(n+1, 1)$ both being induced from fundamental form representations of a parabolic subalgebra. We apply the F -method to derive explicit formulas for such homomorphisms. In particular, we find explicit formulas for the generators of the intertwining operators of the related branching problems restricting generalized Verma modules for $\mathfrak{so}(n+1, 1)$ to $\mathfrak{so}(n, 1)$. As consequences, we find closed formulas for all conformal symmetry breaking differential operators in terms of the first-order operators d , δ , \bar{d} and $\bar{\delta}$ and certain hypergeometric polynomials. A dominant role in these studies will be played by two infinite sequences of symmetry breaking differential operators which depend on a complex parameter λ . These will be termed the conformal first and second type symmetry breaking operators. Their values at special values of λ appear as factors in two systems of factorization identities which involve the Branson-Gover operators of the Euclidean metrics on \mathbb{R}^n and \mathbb{R}^{n-1} and the operators d , δ , \bar{d} and $\bar{\delta}$ as factors, respectively. Moreover, they are shown to naturally recover the gauge companion and Q -curvature operators of the Euclidean metric on the subspace \mathbb{R}^{n-1} , respectively.

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1. INTRODUCTION

The present paper is motivated by recent developments in conformal differential geometry. One of the central notions in this area is Branson's critical Q -curvature $Q_n(g) \in C^\infty(M)$ of a Riemannian manifold (M, g) of even dimension n . The curvature invariant $Q_n(g)$ is the last element in the sequence $Q_2(g), Q_4(g), \dots, Q_n(g)$ of curvature invariants of respective orders $2, 4, \dots, n$ [B95]. If (M, g) is regarded as a hypersurface in the Riemannian manifold (X, g) , the Q -curvatures of (M, g) naturally contribute to the structure of one-parameter families of conformally covariant differential operators

$$D_{2N}(g; \lambda) : C^\infty(X) \rightarrow C^\infty(M), \quad \lambda \in \mathbb{C} \quad (1.1)$$

of order $2N$ mapping functions on the ambient manifold X to functions on M [J09]. Here conformal covariance means that

$$e^{(\lambda+N)\iota^*(\varphi)} D_{2N}(e^{2\varphi}g; \lambda)(u) = D_{2N}(g; \lambda)(e^{\lambda\varphi}u)$$

for all $u, \varphi \in C^\infty(X)$, and ι^* denotes the pull-back defined by the embedding $\iota : M \hookrightarrow X$. Such families generalize the even-order families

$$D_{2N}(\lambda) : C^\infty(S^{n+1}) \rightarrow C^\infty(S^n), \quad \lambda \in \mathbb{C} \quad (1.2)$$

of differential operators which are associated to the equatorial embedding $S^n \hookrightarrow S^{n+1}$ of spheres, and which intertwine spherical principal series representations of the conformal group of the embedded round sphere S^n , but not of the conformal group of the ambient round sphere S^{n+1} . The latter lack of invariance motivates to refer to them as *symmetry breaking operators* [KS13, KP14].

The closely related residue families [J09] are associated to embeddings $\iota : M \rightarrow M \times [0, \varepsilon)$, $\iota(m) = (m, 0)$ of M into small neighborhoods. In this case, the metrics on the neighborhoods are determined by the Poincaré-Einstein metric of the metric g on M in the sense of Fefferman and Graham [FG11]. The connection of residue families to Q -curvatures reveals astonishing recursive structures in the sequence Q_{2N} of Q -curvatures [J13].

The equivariant flat models $C^\infty(\mathbb{R}^{n+1}) \rightarrow C^\infty(\mathbb{R}^n)$ of (1.2) coincide with the residue families for the flat metric on \mathbb{R}^n . In turn, these operators correspond to homomorphisms of generalized Verma modules. Their classification is an algebraic problem [J09], [KOSS15].

In the present paper, we study conformal symmetry breaking differential operators

$$\Omega^p(\mathbb{R}^n) \rightarrow \Omega^q(\mathbb{R}^{n-1})$$

on differential forms which are associated to the embedding $\iota : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$. We shall use Euclidean coordinates x_1, \dots, x_n on \mathbb{R}^n and assume that $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ is the hyperplane $x_n = 0$. The operators of interest are equivariant for the conformal Lie algebra $\mathfrak{so}(n, 1, \mathbb{R})$ of \mathbb{R}^{n-1} with respect to principal series representations of $\mathfrak{so}(n, 1, \mathbb{R})$ and $\mathfrak{so}(n+1, 1, \mathbb{R})$ realized on forms on \mathbb{R}^{n-1} and \mathbb{R}^n , respectively. The operators correspond to symmetry breaking differential operators $\Omega^p(S^n) \rightarrow \Omega^q(S^{n-1})$. We emphasize that the study of symmetry breaking operators of that type which are *not* differential operators remains outside the scope of this paper. However, for the sake of simplicity we shall often refer to conformal symmetry breaking differential operators just as conformal symmetry breaking operators.

A fundamental method for the systematic study of symmetry breaking differential operators was introduced in [KOSS15, K14, KP14] and is termed the F -method. It provides a systematic method to study the corresponding homomorphism of generalized Verma modules. In our situation, it yields a description of the corresponding homomorphisms of generalized Verma modules for the two orthogonal Lie algebras $\mathfrak{so}(n, 1, \mathbb{R})$ and $\mathfrak{so}(n+1, 1, \mathbb{R})$. Applying the F -method, actually leads to explicit descriptions of all conformal symmetry breaking differential operators on differential forms. It turns out that the operators of interest are given by two infinite sequences

$$D_N^{(p \rightarrow p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1}) \quad \text{and} \quad D_N^{(p \rightarrow p-1)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^{n-1}), \quad \mathbb{N} \in \mathbb{N}_0$$

of one-parameter families, some additional operators, and compositions of these with the Hodge star operator of the Euclidean metric on \mathbb{R}^{n-1} . These operators will be referred to as the first and second type families and the third and fourth type operators, respectively. We shall display explicit formulas for all operators and establish their basic mapping properties.

The following theorem is our main result on even-order families of the first type (Theorem 5.2.1).

Theorem 1. *Assume that $N \in \mathbb{N}$ and $0 \leq p \leq n-1$. The one-parameter family*

$$D_{2N}^{(p \rightarrow p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C},$$

of differential operators, which is defined by the formula

$$D_{2N}^{(p \rightarrow p)}(\lambda) = \sum_{i=1}^{N-1} (\lambda + p - 2i) \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^* (\bar{\delta} \bar{d})^i$$

$$+ (\lambda + p) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^* (\bar{d}\bar{\delta})^i + (\lambda + p - 2N) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (\delta d)^{N-i} \iota^* (\bar{\delta}\bar{d})^i \quad (1.3)$$

with the coefficients

$$\alpha_i^{(N)}(\lambda) = (-1)^i 2^N \frac{N!}{(2N)!} \binom{N}{i} \prod_{k=i+1}^N (2\lambda + n - 2k) \prod_{k=1}^i (2\lambda + n - 2k - 2N + 1), \quad (1.4)$$

satisfies the intertwining relation

$$d\pi_{-\lambda+2N-p}^{(p)}(X) D_{2N}^{(p \rightarrow p)}(\lambda) = D_{2N}^{(p \rightarrow p)}(\lambda) d\pi_{-\lambda-p}^{(p)}(X) \quad \text{for all } X \in \mathfrak{so}(n, 1, \mathbb{R}). \quad (1.5)$$

Here we used the following notation. In (1.3), the operators d and δ are the differential and co-differential on forms on \mathbb{R}^{n-1} . Their counterparts on \mathbb{R}^n are denoted by \bar{d} and $\bar{\delta}$. ι^* denotes the pull-back induced by the embedding $\iota : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$. Finally, the representations $\pi_\lambda^{(p)}$ on $\Omega^p(\mathbb{R}^n)$ are defined by

$$\pi_\lambda^{(p)}(\gamma) = e^{\lambda \Phi_\gamma} \gamma_*$$

for all conformal diffeomorphisms γ of the Euclidean metric g_0 on \mathbb{R}^n so that $\gamma_*(g_0) = e^{2\Phi_\gamma} g_0$ for some $\Phi_\gamma \in C^\infty(\mathbb{R}^n)$. The analogous representations on $\Omega^p(\mathbb{R}^{n-1})$ are denoted by $\pi_\lambda'^{(p)}$. The coefficients $\alpha_i^{(N)}(\lambda)$ are related to Jacobi polynomials (Remark 5.1.1).

Note that one may also include (a constant multiple of) the operator ι^* as the case $N = 0$ in the sequence $D_{2N}^{(p \rightarrow p)}(\lambda)$.

By formula (1.3), the families $D_{2N}^{(p \rightarrow p)}(\lambda)$ are *linear* in the degree p of the forms. The family $D_{2N}^{(0 \rightarrow 0)}(\lambda)$ equals the product of $(\lambda - 2N)$ and the equivariant even-order family $D_{2N}(\lambda)$ studied in [J09] and [KOSS15].¹ We recall that the families $D_{2N}(\lambda) : C^\infty(S^n) \rightarrow C^\infty(S^{n-1})$ interpolate between the GJMS-operators of order $2N$ on S^n and S^{n-1} . Similarly, the conformal symmetry breaking operators $D_{2N}^{(p \rightarrow p)}(\lambda)$ interpolate between

$$D_{2N}^{(p \rightarrow p)}(N - \frac{n-1}{2}) = -L_{2N}^{(p)} \iota^* \quad (1.6)$$

and

$$D_{2N}^{(p \rightarrow p)}(N - \frac{n}{2}) = -\iota^* \bar{L}_{2N}, \quad (1.7)$$

where $L_{2N}^{(p)}$ and $\bar{L}_{2N}^{(p)}$ are the respective Branson-Gover operators of order $2N$ on the Euclidean spaces \mathbb{R}^{n-1} and \mathbb{R}^n [BG05]. These identities are special cases of the factorizations of the families $D_{2N}^{(p \rightarrow p)}(\lambda)$ (for specific choices of λ) into products of lower-order families and Branson-Gover operators (Theorem 6.2.1).

The following result provides the analogous formula for odd-order families $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1})$ of the first type (see Theorem 5.3.1 and Remark 5.3.2).

Theorem 2. *Assume that $N \in \mathbb{N}_0$ and $0 \leq p \leq n - 1$. The one-parameter family*

$$D_{2N+1}^{(p \rightarrow p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C},$$

¹The families $D_{2N}^{(p \rightarrow p)}(\lambda)$ raise conformal weights. In contrast, all families in [J09] lower conformal weights because of opposite conventions for representations. The conventions here correspond to those in [KOSS15].

of differential operators, which is defined by the formula

$$\begin{aligned} D_{2N+1}^{(p \rightarrow p)}(\lambda) &= \sum_{i=1}^N \gamma_i^{(N)}(\lambda; p) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta} \bar{d})^i \\ &\quad + (\lambda + p) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d} \bar{\delta})^i \\ &\quad + (\lambda + p - 2N - 1) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d} (\bar{\delta} \bar{d})^i \end{aligned} \quad (1.8)$$

with the coefficients

$$\beta_i^{(N)}(\lambda) \stackrel{\text{def}}{=} (-1)^i 2^N \frac{N!}{(2N+1)!} \binom{N}{i} \prod_{k=i+1}^N (2\lambda + n - 2k) \prod_{k=1}^i (2\lambda + n - 2k - 2N - 1) \quad (1.9)$$

and

$$\gamma_i^{(N)}(\lambda; p) = (\lambda + p - 2i) \beta_i^{(N)}(\lambda) - (\lambda + p - 2i + 1) \beta_{i-1}^{(N)}(\lambda), \quad (1.10)$$

satisfies the intertwining relation

$$d\pi'_{-\lambda+2N+1-p}(X) D_{2N+1}^{(p \rightarrow p)}(\lambda) = D_{2N+1}^{(p \rightarrow p)}(\lambda) d\pi_{-\lambda-p}^{(p)}(X) \quad \text{for all } X \in \mathfrak{so}(n, 1, \mathbb{R}). \quad (1.11)$$

Again, the coefficients $\beta_i^{(N)}(\lambda)$ are related to Jacobi polynomials (Remark 5.1.1). Moreover, ∂_n denotes the normal vector field of the hyperplane \mathbb{R}^{n-1} and i_{∂_n} inserts that field into the forms. For $p = 0$, the family $D_{2N+1}^{(p \rightarrow p)}(\lambda)$ reduces to the product of $(\lambda - 2N - 1)$ and the equivariant odd-order family $D_{2N+1}(\lambda)$ studied in [J09].

The analogous formulas for the conformal symmetry breaking families of the second type follow from the above results for the families of the first type by *conjugation* with the Hodge star operators on \mathbb{R}^n and \mathbb{R}^{n-1} (Theorem 4.3.3 and Theorems 5.2.2, 5.3.3). These operators generalize the operator $\iota^* i_{\partial_n} : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^{n-1})$.

For general form degrees p , the families in Theorem 1 and Theorem 2 are uniquely determined by their equivariance, up to scalar multiples. However, on middle degree forms $p = \frac{n}{2}$ if n is even and $p = \frac{n-1}{2}$ if n is odd, there are additional equivariant families. In fact, if n is even, additional conformal symmetry breaking operators

$$\Omega^{\frac{n}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n}{2}}(\mathbb{R}^{n-1}) \quad \text{and} \quad \Omega^{\frac{n}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n}{2}-1}(\mathbb{R}^{n-1})$$

are given by the respective compositions

$$D_N^{(\frac{n}{2} \rightarrow \frac{n}{2})}(\lambda) \bar{\star} \quad \text{and} \quad D_N^{(\frac{n}{2} \rightarrow \frac{n}{2}-1)}(\lambda) \bar{\star}, \quad (1.12)$$

where $\bar{\star}$ denotes the Hodge star operator of the Euclidean metric on \mathbb{R}^n . Similarly, for odd n , additional conformal symmetry breaking operators

$$\Omega^{\frac{n-1}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \quad \text{and} \quad \Omega^{\frac{n+1}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n-1}{2}}(\mathbb{R}^{n-1})$$

are given by the respective compositions

$$\star D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2})}(\lambda) \quad \text{and} \quad \star D_N^{(\frac{n+1}{2} \rightarrow \frac{n-1}{2})}(\lambda), \quad (1.13)$$

where \star is the Hodge star operator of the Euclidean metric on \mathbb{R}^{n-1} .

For general form degrees, the analogous *compositions* of families of the first and the second type with Hodge star operators yield further equivariant differential operators on

differential forms. The following classification result describes all such symmetry breaking differential operators.

Theorem 3. *The linear differential operators*

$$D : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^q(\mathbb{R}^{n-1})$$

which satisfy the intertwining relation

$$d\pi'_\eta(X)D = Dd\pi_\mu^{(p)}(X) \quad (1.14)$$

for all $X \in \mathfrak{so}(n, 1, \mathbb{R})$ and some $\mu, \eta \in \mathbb{C}$ are generated by the following operators.

- (1) *The case $q = p$. The families $D_N^{(p \rightarrow p)}(\lambda)$ of the first type. Here $N \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and $\mu = -\lambda - p$, $\eta = -\lambda - p + N$.*
- (2) *The case $q = p - 1$. The families $D_N^{(p \rightarrow p-1)}(\lambda)$ of the second type. Here $N \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and $\mu = -\lambda - p$, $\eta = -\lambda - (p - 1) + N$.*
- (3) *The case $q = 1$ and $p = 0$. The operators $d\dot{D}_N^{(0 \rightarrow 0)}(N)$ with $N \in \mathbb{N}_0$. Here $\mu = -N$ and $\eta = 0$.*
- (4) *The case $q = p + 1$. The operator $d\iota^*$. Here $\mu = \eta = 0$.*
- (5) *The case $q = n - 2$ and $p = n$. The operators $\delta\dot{D}_N^{(n \rightarrow n-1)}(N)$ with $N \in \mathbb{N}_0$. Here $\mu = -n - N$ and $\eta = -n + 3$.*
- (6) *The case $q = p - 2$. The operator $\delta\iota^*i_{\partial_n}$. Here $\mu = n - 2p$ and $\eta = n - 2p + 3$.*
- (7) *The compositions of the above operators with \star .*

In (3) and (5), the dot denotes derivatives of the families with respect to the parameter λ .

Some comments concerning this classification are in order.

Firstly, we observe that $dD_N^{(p \rightarrow p)}(N - p) = 0$ and that the derivative of the composition $dD_N^{(p \rightarrow p)}(\lambda)$ (with respect to the parameter λ) at $\lambda = N - p$ is equivariant only if $p = 0$.² This yields the operators in Theorem 3/(3). Similarly, $\delta D_N^{(p \rightarrow p-1)}(N - n + p) = 0$ and the derivative of the composition $\delta D_N^{(p \rightarrow p-1)}(\lambda)$ at $\lambda = N - n + p$ is equivariant only if $p = n$.³ This yields the operators in Theorem 3/(5).

There are some overlaps in the classification: Indeed, for $N = 0$, (3) is a special case of (4), and (5) is a special case of (6).

Theorem 4.3.3 implies that the operators in Theorem 3/(1),(2), Theorem 3/(3),(5) and Theorem 3/(4),(6) are Hodge-conjugate to each other.

The classification implies that all conformal symmetry breaking operators of the type

$$\Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1})$$

are given by the families

$$D_N^{(p \rightarrow p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1})$$

and

$$\star D_N^{(\frac{n}{2} \rightarrow \frac{n}{2}-1)}(\lambda) : \Omega^{\frac{n}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n}{2}}(\mathbb{R}^{n-1})$$

²Only for the degree $p = 0$ the first type families $D_N^{(p \rightarrow p)}(\lambda)$ are given by one sum. They have the property $D_N^{(0 \rightarrow 0)}(N) = 0$.

³Only for the degree $p = n$, the second type families $D_N^{(p \rightarrow p-1)}(\lambda)$ are given by one sum. They have the property $D_N^{(n \rightarrow n-1)}(N) = 0$.

for even n and

$$\star D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2})}(\lambda) : \Omega^{\frac{n-1}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n-1}{2}}(\mathbb{R}^{n-1})$$

for odd n as well as the additional operators

$$\begin{aligned} \star d\iota^* &: \Omega^{\frac{n}{2}-1}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n}{2}-1}(\mathbb{R}^{n-1}), \\ \star \delta \iota^* i_{\partial_n} &: \Omega^{\frac{n+1}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n+1}{2}}(\mathbb{R}^{n-1}). \end{aligned}$$

Since the second family is proportional to $D_N^{(\frac{n}{2} \rightarrow \frac{n}{2})}(\lambda) \bar{\star}$, it decomposes as the direct sum of two families which are given by $D_N^{(\frac{n}{2} \rightarrow \frac{n}{2})}(\lambda)$. An analogous comment concerns the third family. In addition, for $n = 2$, we have the operators

$$\star d\dot{D}_N^{(0 \rightarrow 0)}(N) : \Omega^0(\mathbb{R}^2) \rightarrow \Omega^0(\mathbb{R}^1).$$

Similarly, it follows that all conformal symmetry breaking operators of the type

$$\Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p+1}(\mathbb{R}^{n-1})$$

are given by the families

$$\star D_N^{(\frac{n}{2}-1 \rightarrow \frac{n}{2}-1)}(\lambda) : \Omega^{\frac{n}{2}-1}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n}{2}}(\mathbb{R}^{n-1})$$

for even n and

$$\star D_N^{(\frac{n-1}{2} \rightarrow \frac{n-3}{2})}(\lambda) : \Omega^{\frac{n-1}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n+1}{2}}(\mathbb{R}^{n-1})$$

for odd n as well as the additional operators

$$\begin{aligned} d\iota^* &: \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p+1}(\mathbb{R}^{n-1}), \\ d\dot{D}_N^{(0 \rightarrow 0)}(N) &: \Omega^0(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}^{n-1}) \end{aligned}$$

and

$$\star d\dot{D}_N^{(0 \rightarrow 0)}(N) : \Omega^0(\mathbb{R}^3) \rightarrow \Omega^1(\mathbb{R}^2).$$

In dimension $n = 2$, the conformal symmetry breaking operators $D_N^{(1,0)}(\lambda)$ and $D_N^{(1,0)}(\lambda) \bar{\star}$ were found in [KKP14] (see Remark 4.8.8).

We continue with the description of two basic properties of the families of the first and second type.

Firstly, we note that both types of conformal symmetry breaking operators are connected through systems of additional factorization identities which involve the operators d , δ , \bar{d} and $\bar{\delta}$ as factors (Theorems 6.3.1 and 6.3.2). The simplest of these relations are

$$-(2N)dD_{2N-1}^{(p \rightarrow p-1)}(-p+2N) = D_{2N}^{(p \rightarrow p)}(-p+2N) \quad (1.15)$$

and

$$(2N)D_{2N-1}^{(p+1 \rightarrow p)}(-p-1)\bar{d} = D_{2N}^{(p \rightarrow p)}(-p). \quad (1.16)$$

Note that the latter identities imply that $dD_{2N}^{(p \rightarrow p)}(-p+2N) = 0$ and $D_{2N}^{(p \rightarrow p)}(-p)\bar{d} = 0$.

The following result states respective connections of both types of conformal symmetry breaking operators to the critical Q -curvature operators $Q_{n-1-2p}^{(p)}$ and the gauge companion operators $G_{n-2p}^{(p)}$ of (\mathbb{R}^{n-1}, g_0) (we refer to Section 6.1 for the definition of these concepts). Let dot denote the derivative with respect to λ .

Theorem 4. *If $n - 1$ is even and $n - 2p \geq 3$, we have*

$$\dot{D}_{n-1-2p}^{(p \rightarrow p)}(-p)|_{\ker(\bar{d})} = Q_{n-1-2p}^{(p)} \iota^*. \quad (1.17)$$

Similarly, if $n - 1$ is even and $n - 2p \geq 1$, then

$$D_{n-2p}^{(p \rightarrow p-1)}(-p)|_{\ker(\bar{d})} = -G_{n-2p}^{(p)} \iota^*. \quad (1.18)$$

Although the Q -curvature $Q_n = Q_n^{(0)}$ of the Euclidean metric on \mathbb{R}^n vanishes, the formula (1.17) is non-trivial. However, it is an easy consequence of formula (1.3) for the even-order families of the first type. An important aspect of formula (1.17) is the fact that it resembles its analog

$$\dot{D}_n^{res}(0; g)(1) = Q_n(g)$$

for general metrics g on a manifold of even dimension n [J09], [GJ07]. For more details see Theorem 6.4.1 and Theorem 6.4.8.

We finish this section with an outline of the content of the paper.

In Section 2, we first review an application of the Fourier transform to generalized Verma modules which is known as the F -method. We explain the role of singular vectors in this connection and describe qualitative results on some branching laws.

Section 3 is devoted to the construction of singular vectors which correspond to conformal symmetry breaking operators $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^q(\mathbb{R}^{n-1})$. We find two one-parameter families of singular vectors which describe the embedding of the $\mathfrak{so}(n, 1, \mathbb{R})$ -submodules in the branching laws of the generalized Verma modules which are induced from twisted fundamental representations on exterior forms. They are termed the singular vectors of first and second type. Their construction rests on solving systems of ordinary differential equations produced by the F -method. It turns out that the solution spaces are determined by Gegenbauer polynomials. The main results are Theorems 3.2.1, 3.2.2, 3.3.1 and 3.3.2. Finally, reducible summands in the branching laws lead to additional singular vectors of the third and the fourth type (Theorem 3.4.1, Theorem 3.5.1). There are some subtleties for middle degree forms.

Then, in Section 4, we apply the results on singular vectors to define conformal symmetry breaking operators. The singular vectors of the first type induce the first type families $D_N^{(p \rightarrow p)}(\lambda)$ of conformal symmetry breaking operators of order $N \in \mathbb{N}$ acting on differential p -forms on \mathbb{R}^n and taking values in p -forms on \mathbb{R}^{n-1} . By conjugation with the Hodge star operators on \mathbb{R}^n and \mathbb{R}^{n-1} , we obtain the second type families $D_N^{(p \rightarrow p-1)}(\lambda)$ of conformal symmetry breaking operators. They correspond to the singular vectors of the second type found in Section 3 and act on differential p -forms on \mathbb{R}^n and take values in $(p-1)$ -forms on \mathbb{R}^{n-1} . The main results here are Theorems 4.1.1 and 4.1.2 (first type families) and Theorems 4.2.1 and 4.2.2 (second type families). Next, singular vectors of type three and four induce conformal symmetry breaking operators of type three and four, respectively. Again, these two types are Hodge conjugate to each other. The main results here are Theorems 4.4.1 and 4.4.2 (type three) and Theorems 4.5.1 and 4.5.2 (type four). Finally, in Section 4.7, we combine the previous results in a proof of the classification result Theorem 3. In the last two subsections of Section 4, we display explicit formulas for low-order special cases, confirm the equivariance of the first-order families both by direct calculations and as consequences of the conformal covariance of their curved analogs and prove that the results of [KKP14] in dimension $n = 2$ are special cases of ours.

In Section 5, we derive alternative closed formulas for all conformal symmetry breaking operators in terms of the geometric operators d , δ and their bar-versions. That process amounts to changing to another basis of invariants. Although the resulting formulas allow quick proofs of some of the basic properties of the families in later sections, it should be stressed that neither of the two ways to write them is canonical. The discussion of the first and second type families is divided into two subsections according to the parity of their orders. The main results are contained in Theorems 5.2.1 and 5.2.2 (even order) and Theorems 5.3.1 and 5.3.3 (odd order). These results cover Theorem 1 and Theorem 2. An analogous discussion yields geometric formulas for operators of type three and four.

In Section 6, we establish basic properties of the first and second type families of conformal symmetry breaking operators. We first recall some results on Branson-Gover, gauge companion and Q -curvature operators [BG05] on forms. Then we derive systems of identities which show that at special arguments any family of the first type factors into the product of a lower-order family of the first type and a Branson-Gover operator for the Euclidean metrics on \mathbb{R}^n or \mathbb{R}^{n-1} (Theorems 6.2.1 and 6.2.2). The special arguments for which such *main factorizations* take place are naturally determined by the conformal weights of the Branson-Gover operators involved. As very special cases, we obtain the relations (1.6) and (1.7). Furthermore, we show that both types of conformal symmetry breaking operators are linked by systems of *supplementary factorizations* (Theorems 6.3.1 and 6.3.2) which involve the four operators d , δ , \bar{d} and $\bar{\delta}$ as factors. The overall interest in both types of factorization identities comes from the conjecture that they literally extend to curved analogs of the families. Theorem 6.4.1 (see also (1.18)) gives a description of the gauge companion operators for the Euclidean metric in terms of odd-order conformal symmetry breaking operators of the second type. Finally, for the Euclidean metric, we introduce the notion of Q -curvature polynomials (Definition 6.4.3) on differential forms (generalizing the case $p = 0$ analyzed in [J13]) and show how these cover the Q -curvature operator on closed differential forms (see Theorem 6.4.6).

The results of the present paper solve a basic problem of conformal geometry in a very special but important case. Indeed, it is natural to ask for a description of all *conformally covariant* differential operators mapping differential forms on a Riemannian manifold (X, g) to differential forms on a submanifold (M, g) (with the induced metric).⁴ In the codimension one case and for differential operators acting on functions, that problem has been analyzed in [J09]. The case of differential operators on forms on $(X, M) = (\mathbb{R}^n, \mathbb{R}^{n-1})$ with the Euclidean metric on the background manifold \mathbb{R}^n is studied here. The significance of the present results for the general curved case comes from the fact that the formulas in the curved case arise from the geometric formulas derived here by adding lower-order *curvature correcting terms* which involve curvature of the submanifold and curvature being associated to the embedding. However, closed formulas for these correction terms seem to be out of reach presently. There are at least two ways to attempt to understand their structure. In the codimension one case, a purely algebraic approach could be based on the construction of lifts of our families of homomorphisms to families of homomorphisms of semi-holonomic Verma modules [ES97]. A different approach will be developed in the forthcoming paper [FJS16], where we identify the families constructed here with two types of so-called residue family operators on differential forms. For general

⁴A version of the problem appears as Problem 2.2 in [KKP14]. It replaces conformal covariance by equivariance with respect to an appropriate group of conformal diffeomorphisms.

metrics, these residue families are defined by the asymptotic expansions of eigenforms of the Laplacian of a Poincaré-Einstein metric of a conformal manifold. Their construction extends the construction of residue families in [J09].

2. PRELIMINARIES

In the present section, we briefly review basic notation and results initiated and developed in [K12, KP14, KOSS15, K14]. We apply these results to describe the situation of our interest: the branching problem for generalized Verma modules of real orthogonal simple Lie algebras, conformal parabolic subalgebras and fundamental representations on forms twisted by characters as inducing data.

2.1. The F -method. We first introduce our setting. Let G be a connected real reductive Lie group with Lie algebra $\mathfrak{g}(\mathbb{R})$, $P \subset G$ a parabolic subgroup with Lie algebra $\mathfrak{p}(\mathbb{R})$, $\mathfrak{p}(\mathbb{R}) = \mathfrak{l}(\mathbb{R}) \oplus \mathfrak{n}_+(\mathbb{R})$ its Levi decomposition and $\mathfrak{n}_-(\mathbb{R})$ the opposite (negative) nilradical, $\mathfrak{g}(\mathbb{R}) = \mathfrak{n}_-(\mathbb{R}) \oplus \mathfrak{p}(\mathbb{R})$. The complexifications of Lie algebras $\mathfrak{g}(\mathbb{R})$, $\mathfrak{p}(\mathbb{R})$, $\mathfrak{l}(\mathbb{R})$, $\mathfrak{n}_+(\mathbb{R})$, etc., will be denoted \mathfrak{g} , \mathfrak{p} , \mathfrak{l} , \mathfrak{n}_+ , etc.

Given a complex finite dimensional P -module (ρ, V) , we consider the induced representation $(\pi, \text{Ind}_P^G(V))$ of G on smooth sections of the homogeneous vector bundle $G \times_P V \rightarrow G/P$, i.e., π acts on the space

$$\text{Ind}_P^G(V) \stackrel{\text{def}}{=} C^\infty(G, V)^P = \{f \in C^\infty(G, V) \mid f(gp) = \rho(p^{-1})f(g), g \in G, p \in P\}$$

by the left regular representation. Given another data $(G', P', \rho' : P' \rightarrow \text{GL}(V'))$ such that $G' \subset G$ is reductive, $P' = P \cap G'$ and \mathfrak{p} is \mathfrak{g}' -compatible, cf. [KP14, Definition 4.5], the space of continuous G' -equivariant homomorphisms

$$\text{Hom}_{G'} \left(\text{Ind}_P^G(V), \text{Ind}_{P'}^{G'}(V') \right) \quad (2.1)$$

is termed the space of *symmetry breaking operators* [KS13]. Here the G' -equivariance is defined with respect to the actions $\pi(g')$ and $\pi'(g')$ for all $g' \in G'$, where $\pi(g')$ is realized by embedding $G' \subset G$. The space of G' -equivariant differential operators $D : \text{Ind}_P^G(V) \rightarrow \text{Ind}_{P'}^{G'}(V')$, cf. [KP14, Section 2], denoted by

$$\text{Diff}_{G'} \left(\text{Ind}_P^G(V), \text{Ind}_{P'}^{G'}(V') \right), \quad (2.2)$$

is a subspace of (2.1).

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} . Denoting V^\vee the dual (or, contragredient) representation to V , the generalized Verma module $\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ is defined by

$$\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \stackrel{\text{def}}{=} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^\vee.$$

The induced action will be denoted by π^\vee .

The space (2.2) of G' -equivariant differential operators is isomorphic to the space of \mathfrak{g}' -homomorphisms of generalized Verma modules,

$$\text{Diff}_{G'} \left(\text{Ind}_P^G(V), \text{Ind}_{P'}^{G'}(V') \right) \simeq \text{Hom}_{\mathfrak{g}'} \left(\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}((V')^\vee), \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \right), \quad (2.3)$$

where in addition

$$\text{Hom}_{\mathfrak{g}'} \left(\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}((V')^\vee), \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \right) \simeq \text{Hom}_{\mathfrak{p}'} \left((V')^\vee, \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \right), \quad (2.4)$$

cf. [KP14, Theorem 2.7], [KOSS15]. In [KP14], the elements of $\text{Hom}_{\mathfrak{p}'}((V')^\vee, \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$ are referred to as *singular vectors*. Alternatively, we may first define the L' -submodule

$$\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+} \stackrel{\text{def}}{=} \{v \in \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \mid d\pi^\vee(Z)v = 0, Z \in \mathfrak{n}'_+\} \quad (2.5)$$

of $\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ as the *space of singular vectors* and then study the \mathfrak{l}' -homomorphisms of $(V')^\vee$ into this space. Then any irreducible \mathfrak{l}' -submodule $(W')^\vee$ of $\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ yields a \mathfrak{g}' -homomorphism $\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}((W')^\vee)$ to $\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$. The latter method has been used in [KOSS15].

The F -method, cf. [KOSS15], [KP14, Section 4], is a constructive method to determine singular vectors. The idea is the following. We first note that a generalized Verma module $\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ can be realized as a space of V^\vee -valued distributions on N_- supported at $o \stackrel{\text{def}}{=} eP \in G/P$. In particular, there is a $\mathcal{U}(\mathfrak{g})$ -module isomorphism

$$\phi : \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \rightarrow \mathcal{D}'_{[0]}(N_-, V^\vee) \quad (2.6)$$

([KOSS15], [KP14]). When the nilpotent group N_- is commutative, it may be identified with $\mathfrak{n}_-(\mathbb{R})$ via the exponential map. Hence

$$\mathcal{D}'_{[0]}(N_-, V^\vee) \simeq \mathcal{D}'_{[0]}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee.$$

Let $\text{Pol}(\mathfrak{n}_-^*(\mathbb{R}))$ be the space of polynomials on $\mathfrak{n}_-^*(\mathbb{R})$. The algebraic *Fourier transform*⁵

$$\mathcal{F} : \mathcal{D}'_{[0]}(\mathfrak{n}_-(\mathbb{R})) \rightarrow \text{Pol}(\mathfrak{n}_-^*(\mathbb{R})), \quad \mathcal{F}(f)(\xi) \stackrel{\text{def}}{=} \langle f(\cdot), e^{i\langle \cdot, \xi \rangle} \rangle = \int_{\mathfrak{n}_-(\mathbb{R})} e^{i\langle x, \xi \rangle} f(x) dx, \quad (2.7)$$

is an algebra isomorphism mapping convolutions into products. \mathcal{F} extends to an isomorphism of vector spaces

$$\mathcal{F} \otimes \text{Id}_{V^\vee} : \mathcal{D}'_{[0]}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee \rightarrow \text{Pol}(\mathfrak{n}_-^*(\mathbb{R})) \otimes V^\vee. \quad (2.8)$$

We use this isomorphism to transport the \mathfrak{g} -module structure. Let $d\tilde{\pi}$ denote the induced action of \mathfrak{g} on $\text{Pol}(\mathfrak{n}_-^*(\mathbb{R})) \otimes V^\vee$. Thus the space of singular vectors is identified by Fourier transform with the space

$$\text{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee) \stackrel{\text{def}}{=} \{f \in \text{Pol}(\mathfrak{n}_-^*(\mathbb{R})) \otimes V^\vee \mid d\tilde{\pi}(Z)f = 0, Z \in \mathfrak{n}'_+\}. \quad (2.9)$$

In summary, searching for singular vectors, which is a combinatorial problem, is equivalent to finding elements in $\text{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee)$, a problem in algebraic analysis. Finally, we note that the operators $d\tilde{\pi}(Z)$ in (2.9) form a system of *second-order* partial differential operators on $\text{Pol}(\mathfrak{n}_-^*(\mathbb{R})) \otimes V^\vee$.

2.2. Notation and induced representations. In the remainder of the paper, we assume that $N \ni n \geq 2$. Let $G = SO_0(n+1, 1, \mathbb{R})$ be the connected component of the identity of the group preserving the quadratic form

$$2x_0x_{n+1} + x_1^2 + \cdots + x_n^2$$

on \mathbb{R}^{n+2} . Let $\{e_j, j = 0, \dots, n+1\}$ be the standard basis of \mathbb{R}^{n+2} . The group G preserves the cone

$$\mathcal{C} = \{x = (x_0, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid 2x_0x_{n+1} + x_1^2 + \cdots + x_n^2 = 0\}.$$

⁵By an artificial use of the pairing of \mathfrak{n}_+ and \mathfrak{n}_- defined by the Killing form, one may regard the range of the Fourier transform as consisting of polynomials on \mathfrak{n}_+ . Although this has been the perspective in [KOSS15], we prefer not to follow it here. It would also complicate formulas by unpleasant constants.

Let $P = P_+ \subset G$ and $P_- \subset G$ be the isotropy subgroups of the lines generated by $e_0 = (1, 0, \dots, 0)$ and $e_{n+1} = (0, 0, \dots, 1)$, respectively. Then P_\pm are parabolic subgroups with Langlands decompositions $P_\pm = LN_\pm = MAN_\pm$, where $M \simeq SO(n, \mathbb{R})$, $A \simeq \mathbb{R}^+$ and $N_\pm \simeq \mathbb{R}^n$. The elements of G can be written in the block form

$$\begin{pmatrix} 1 \times 1 & 1 \times n & 1 \times 1 \\ n \times 1 & n \times n & n \times 1 \\ 1 \times 1 & 1 \times n & 1 \times 1 \end{pmatrix}$$

with respect to the decomposition

$$\mathbb{R}^{n+2} = \mathbb{R}e_0 \oplus \bigoplus_{j=1}^n \mathbb{R}e_j \oplus \mathbb{R}e_{n+1}.$$

The real Lie algebras corresponding to the above Lie groups are

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{so}(n+1, 1, \mathbb{R}), \quad \mathfrak{p}_\pm(\mathbb{R}) = \mathfrak{l}(\mathbb{R}) \oplus \mathfrak{n}_\pm(\mathbb{R}) = \mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}_\pm(\mathbb{R}) \quad (2.10)$$

with

$$\mathfrak{m}(\mathbb{R}) \simeq \mathfrak{so}(n, \mathbb{R}), \quad \mathfrak{a}(\mathbb{R}) \simeq \mathbb{R} \quad \text{and} \quad \mathfrak{n}_\pm(\mathbb{R}) \simeq \mathbb{R}^n.$$

The basis elements

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_j^+ = \begin{pmatrix} 0 & e_j & 0 \\ 0 & 0 & -e_j^t \\ 0 & 0 & 0 \end{pmatrix}, \quad E_j^- = \begin{pmatrix} 0 & 0 & 0 \\ e_j^t & 0 & 0 \\ 0 & -e_j & 0 \end{pmatrix}, \quad 1 \leq j \leq n$$

and

$$M_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (M_{ij})_{rs} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i, j, r, s = 1, \dots, n, \quad i < j$$

satisfy the commutation relations

$$[E_i^+, E_j^-] = \delta_{ij}E + M_{ij}$$

and

$$[M_{ij}, E_r^+] = \delta_{jr}E_r^+ - \delta_{ir}E_j^+, \quad [E, E_j^+] = E_j^+, \quad [E, M_{ij}] = 0$$

and analogously for E_j^- . We shall also use the notation $E_j = E_j^-$.

The elements E_j^\pm , $1 \leq j \leq n$, form respective bases of $\mathfrak{n}_\pm(\mathbb{R})$. We use these bases to identify both spaces with \mathbb{R}^n . The Euclidean metric g_0 on \mathbb{R}^n induces scalar products on $\mathfrak{n}_\pm(\mathbb{R})$ for which both bases are orthonormal. The adjoint action of M on $\mathfrak{n}_\pm(\mathbb{R})$ preserves these scalar products. Hence M can be identified with $SO(n, \mathbb{R})$. Sometimes it will be useful to identify elements of $\mathfrak{n}_-(\mathbb{R})$ with column vectors and elements of $\mathfrak{n}_+(\mathbb{R})$ with row vectors. Let $(E_j^\pm)^*$ for $1 \leq j \leq n$ be the dual basis elements of $\mathfrak{n}_\pm(\mathbb{R})^*$. They are orthonormal with respect to corresponding dual scalar products.

Furthermore, we consider the subgroup $G' \simeq SO_0(n, 1, \mathbb{R})$ of G which preserves the line in the cone \mathcal{C} generated by e_n . The embedding of G' is realized as the matrix block inclusion

$$\begin{pmatrix} 1 \times 1 & 1 \times n-1 & 1 \times 1 \\ n-1 \times 1 & n-1 \times n-1 & n-1 \times 1 \\ 1 \times 1 & 1 \times n-1 & 1 \times 1 \end{pmatrix} \hookrightarrow \begin{pmatrix} 1 \times 1 & 1 \times n-1 & 0 & 1 \times 1 \\ n-1 \times 1 & n-1 \times n-1 & 0 & n-1 \times 1 \\ 0 & 0 & 0 & 0 \\ 1 \times 1 & 1 \times n-1 & 0 & 1 \times 1 \end{pmatrix}.$$

The group $P' = P \cap G'$ is a parabolic subgroup of G' with Langlands decomposition $P' = L'N'_+ = M'AN'_+$, where $M' \simeq SO(n-1, \mathbb{R})$, $A \simeq \mathbb{R}^+$ and $N'_+ \simeq \mathbb{R}^{n-1}$. The real Lie algebras of these groups are

$$\mathfrak{g}'(\mathbb{R}) = \mathfrak{so}(n, 1, \mathbb{R}), \quad \mathfrak{p}'(\mathbb{R}) = \mathfrak{l}'(\mathbb{R}) \oplus \mathfrak{n}'_+(\mathbb{R}) = \mathfrak{m}'(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}'_+(\mathbb{R}) \quad (2.11)$$

with

$$\mathfrak{m}'(\mathbb{R}) \simeq \mathfrak{so}(n-1, \mathbb{R}), \quad \mathfrak{a}(\mathbb{R}) \simeq \mathbb{R} \quad \text{and} \quad \mathfrak{n}'_+(\mathbb{R}) \simeq \mathbb{R}^{n-1}.$$

The elements E_j^\pm for $1 \leq j \leq n-1$ form respective bases of $\mathfrak{n}'_\pm(\mathbb{R})$.

The natural action of $M \simeq SO(n, \mathbb{R})$ on \mathbb{R}^n induces representations σ_p on the spaces $\Lambda^p(\mathbb{R}^n)$ and $\Lambda^p(\mathbb{R}^n)^*$ of multilinear forms.

Remark 2.2.1. For $X, Y \in \mathbb{R}^n$, the tensor $X \otimes Y - Y \otimes X$ acts on \mathbb{R}^n with the Euclidean metric g_0 by

$$T : v \mapsto Xg_0(Y, v) - Yg_0(X, v).$$

In view of $g_0(T(v), w) + g_0(v, T(w)) = 0$, we have $T \in \mathfrak{so}(n, \mathbb{R})$. The action T naturally extends to $\Lambda^p(\mathbb{R}^n)$ by

$$(X \otimes Y - Y \otimes X)(v_1 \wedge \cdots \wedge v_p) = \sum_{l=1}^p v_1 \wedge \cdots \wedge v_{l-1} \wedge (Xg_0(Y, v_l) - Yg_0(X, v_l)) \wedge v_{l+1} \wedge \cdots \wedge v_p$$

for $v_1 \wedge \cdots \wedge v_p \in \Lambda^p(\mathbb{R}^n)$. It coincides with the infinitesimal representation $d\sigma_p$ on $\Lambda^p(\mathbb{R}^n)$.

We shall use the notation σ_p also for the corresponding representations of $M \simeq SO(n, \mathbb{R})$ on the complex vector spaces $\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}$.

Next, we fix some notation. We define the fundamental weights

$$\begin{aligned} \Lambda_j &= e_1 + \cdots + e_j & \text{for } j = 1, \dots, \frac{n}{2} - 2, \\ \Lambda_{\frac{n}{2}-1} &= \frac{1}{2}(e_1 + \cdots + e_{\frac{n}{2}-1} - e_{\frac{n}{2}}) \\ \Lambda_{\frac{n}{2}} &= \frac{1}{2}(e_1 + \cdots + e_{\frac{n}{2}-1} + e_{\frac{n}{2}}) \end{aligned} \quad (2.12)$$

for even n and

$$\begin{aligned} \Lambda_j &= e_1 + \cdots + e_j & \text{for } j = 1, \dots, \frac{n-3}{2}, \\ \Lambda_{\frac{n-1}{2}} &= \frac{1}{2}(e_1 + \cdots + e_{\frac{n-1}{2}}) \end{aligned} \quad (2.13)$$

for odd n . Here the vectors e_j denote respective standard basis vectors in $\mathbb{R}^{\frac{n-1}{2}}$ and $\mathbb{R}^{\frac{n}{2}}$. Let $\Lambda_0 = 0$. We also set

$$1_p \stackrel{\text{def}}{=} (\underbrace{1, \dots, 1}_{p \text{ entries}}, 0, \dots, 0) \quad \text{and} \quad 1_{\frac{n}{2}}^\pm \stackrel{\text{def}}{=} (\underbrace{1, \dots, 1}_{\frac{n}{2} \text{ entries}}, \pm 1)$$

for even n and

$$1_p \stackrel{\text{def}}{=} (\underbrace{1, \dots, 1}_{p \text{ entries}}, 0, \dots, 0) \quad \text{and} \quad 1_{\frac{n-1}{2}} = (\underbrace{1, \dots, 1}_{\frac{n-1}{2} \text{ entries}}).$$

for odd n .

The following lemma collects basic information on the representations σ_p .

Lemma 2.2.2. (i) If n is odd, the representation σ_p of $SO(n, \mathbb{R})$ on $\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}$ is irreducible for all $p = 0, 1, \dots, n$. σ_p is equivalent to σ_{n-p} . For $p \leq (n-1)/2$, its highest weight equals 1_p .

(ii) If n is even and $p \neq \frac{n}{2}$, the representation σ_p of $SO(n, \mathbb{R})$ on $\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}$ is irreducible. σ_p is equivalent to σ_{n-p} . For $p \leq n/2 - 1$, its highest weight equals 1_p .

(iii) If n is even, we have the decomposition $\sigma_{\frac{n}{2}} = \sigma_{\frac{n}{2}}^+ \oplus \sigma_{\frac{n}{2}}^-$, where the irreducible representations $\sigma_{\frac{n}{2}}^\pm$ act on the eigenspaces

$$\Lambda_{\pm}^{\frac{n}{2}}(\mathbb{R}^n) \otimes \mathbb{C} \stackrel{\text{def}}{=} \{\omega \in \Lambda^{\frac{n}{2}}(\mathbb{R}^n) \otimes \mathbb{C} \mid \bar{\star}\omega = \mu_{\pm}\omega\}$$

of the Hodge star operator $\bar{\star}$ of the Euclidean metric g_0 on \mathbb{R}^n . Here $\mu_{\pm} = \pm 1$ if $n \equiv 0 \pmod{4}$ and $\mu_{\pm} = \pm i$ if $n \equiv 2 \pmod{4}$. The highest weights of the representations $\sigma_{\frac{n}{2}}^+$ and $\sigma_{\frac{n}{2}}^-$ are $1_{\frac{n}{2}}^+$ and $1_{\frac{n}{2}}^-$, respectively.

(iv) For all $p = 0, 1, \dots, n$, there is an isomorphism of $SO(n-1, \mathbb{R})$ -modules

$$\Lambda^p(\mathbb{R}^n) \simeq \Lambda^p(\mathbb{R}^{n-1}) \oplus \Lambda^{p-1}(\mathbb{R}^{n-1});$$

here we set $\Lambda^{-1}(\mathbb{R}^{n-1}) = 0$.

(v) If n is even, there are isomorphisms

$$\Lambda_{+}^{\frac{n}{2}}(\mathbb{R}^n) \otimes \mathbb{C} \simeq \Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C} \quad \text{and} \quad \Lambda_{-}^{\frac{n}{2}}(\mathbb{R}^n) \otimes \mathbb{C} \simeq \Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C},$$

which are given by

$$\Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C} \ni \omega \mapsto \begin{cases} \star\omega + \omega \wedge e_n & \text{if } n \equiv 0 \pmod{4} \\ \star\omega - i\omega \wedge e_n & \text{if } n \equiv 2 \pmod{4} \end{cases}$$

and

$$\Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C} \ni \omega \mapsto \begin{cases} \star\omega - \omega \wedge e_n & \text{if } n \equiv 0 \pmod{4} \\ \star\omega + i\omega \wedge e_n & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Here \star denotes the Hodge star operator (of the Euclidean metric) on \mathbb{R}^{n-1} . In particular, the highest weights of the $SO(n-1, \mathbb{R})$ -modules $\Lambda_{\pm}^{\frac{n}{2}}(\mathbb{R}^n)$ are $1_{\frac{n}{2}-1}$.

Proof. The claims in (i), (ii) and (iii) are well-known; see also Proposition 3.1.1. Note that the $SO(n, \mathbb{R})$ -invariance of the Hodge star operator implies the equivalencies in (i) and (ii), and shows that the eigenspaces in (iii) are well-defined. (iv) follows from the $SO(n-1, \mathbb{R})$ -equivariance of the decomposition

$$\omega = \omega' + \omega'' \wedge e_n \quad \text{with } \omega' \in \Lambda^p(\mathbb{R}^{n-1}) \text{ and } \omega'' \in \Lambda^{p-1}(\mathbb{R}^{n-1})$$

of any $\omega \in \Lambda^p(\mathbb{R}^n)$. In order to prove (v), we first observe that

$$\bar{\star}(\star\omega) = \omega \wedge e_n \quad \text{and} \quad \bar{\star}(\omega \wedge e_n) = (-1)^{\frac{n}{2}} \star\omega$$

for $\omega \in \Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1})$. Then we obtain

$$\bar{\star}(\star\omega \pm \omega \wedge e_n) = \pm(\star\omega \pm \omega \wedge e_n)$$

if $n \equiv 0 \pmod{4}$ and

$$\bar{\star}(\star\omega \pm i\omega \wedge e_n) = \mp i(\star\omega \pm i\omega \wedge e_n)$$

if $n \equiv 2 \pmod{4}$. The proof is complete. \square

Next, we introduce some notation for induced representations. For $\lambda \in \mathbb{C}$, we denote by $(\xi_\lambda, \mathbb{C}_\lambda)$ the 1-dimensional complex representation $\xi_\lambda(a) = \exp(\lambda \log a)$ of A . We trivially extend ξ_λ to a character of P . Its dual representation is $\mathbb{C}_\lambda^\vee \simeq \mathbb{C}_{-\lambda}$. For $p = 0, 1, \dots, n$ and $\lambda \in \mathbb{C}$, we define the representation

$$\rho_{\lambda,p} \stackrel{\text{def}}{=} \sigma_p \otimes \xi_\lambda$$

of $L = MA$ on

$$V_{\lambda,p} \stackrel{\text{def}}{=} \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda \simeq \Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}_\lambda. \quad (2.14)$$

Here we regard the spaces $\Lambda^p(\mathbb{R}^n)$ as trivial A -modules. We also recall that for even n the representation $\sigma_{\frac{n}{2}}$ is *not* irreducible. The dual representation $\rho_{\lambda,p}^\vee$ of $L = MA$ acts on

$$V_{\lambda,p}^\vee \simeq \Lambda^p(\mathfrak{n}_-(\mathbb{R}))^* \otimes \mathbb{C}_{-\lambda} \simeq \Lambda^p(\mathbb{R}^n)^* \otimes \mathbb{C}_{-\lambda}.$$

We trivially extend $\rho_{\lambda,p}$ and $\rho_{\lambda,p}^\vee$ to representations of P , i.e.,

$$\rho_{\lambda,p}(\text{man})(v \otimes 1) = \sigma_p(m)(v) \otimes a^\lambda \quad (2.15)$$

and

$$\rho_{\lambda,p}^\vee(\text{man})(v \otimes 1) = \sigma_p^\vee(m)(v) \otimes a^{-\lambda} \quad (2.16)$$

for $\text{man} \in P = MAN_+$. Let

$$(\pi_{\lambda,p}, \text{Ind}_P^G(\rho_{\lambda,p})) \quad \text{and} \quad (\pi_{\lambda,p}^\vee, \text{Ind}_P^G(\rho_{\lambda,p}^\vee))$$

be the resulting induced representations of G .⁶ The analogous induced representations for G' are denoted by $\pi'_{\lambda,p}$ and $\pi'^\vee_{\lambda,p}$.

Remark 2.2.3. *The representation $\pi_{\lambda,p}^\vee$ can be naturally identified with the left regular representation of G on the space $\Omega^p(G/P, \mathbb{C}_{-\lambda-p})$ of weighted p -forms on G/P (with weight $-\lambda-p$). Alternatively, we may naturally identify the induced representations $\pi_{\lambda,p}^\vee$ with the geometrically defined representations*

$$\pi_\lambda^{(p)}(\gamma) \stackrel{\text{def}}{=} e^{\lambda \Phi_\gamma} \gamma_* : \Omega^p(S^n) \rightarrow \Omega^p(S^n), \quad \gamma \in G \quad (2.17)$$

on complex-valued p -forms on the round sphere S^n through the identity

$$\pi_{-\lambda-p}^{(p)} = \pi_{\lambda,p}^\vee. \quad (2.18)$$

The definition (2.17) rests on the fact that G acts on $S^n = G/P$ by conformal diffeomorphisms of the round metric g_0 , i.e., $\gamma_*(g_0) = e^{2\Phi_\gamma} g_0$ for some $\Phi_\gamma \in C^\infty(S^n)$. For more details on these identifications in the case $p = 0$ see Section 2.3 in [J09].

2.3. A branching problem. In the present section, we use a result of [K12] to describe the decompositions of the generalized Verma modules

$$\mathcal{M}_\mathfrak{p}^\mathfrak{g}(\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}_\lambda), \quad \lambda \in \mathbb{C}$$

for \mathfrak{g} under restriction to the subalgebra \mathfrak{g}' on the level of characters.

Let $S(V) = \bigoplus_{N=0}^\infty S^N(V)$ be the symmetric tensor algebra over the vector space V . We extend the adjoint action of \mathfrak{l}' to $S(\mathfrak{n}_-/\mathfrak{n}'_-)$. Then the \mathfrak{l}' -module $S(\mathfrak{n}_-/\mathfrak{n}'_-)$ is the free commutative ring generated by the 1-dimensional \mathfrak{l}' -module $\mathfrak{n}_-/\mathfrak{n}'_-$ isomorphic to \mathbb{C}_{-1} , i.e.,

⁶In [J09], induced representations are defined by using the opposite sign of λ .

$S(\mathfrak{n}_-/\mathfrak{n}'_-) \simeq \oplus_{N \geq 0} \mathbb{C}_{-N}$. For any finite dimensional irreducible \mathfrak{l} -module W and any finite dimensional irreducible \mathfrak{l}' -module V' we define

$$m(W, V') \stackrel{\text{def}}{=} \dim_{\mathbb{C}} \text{Hom}_{\mathfrak{l}'}(V', W|_{\mathfrak{l}'} \otimes S(\mathfrak{n}_-/\mathfrak{n}'_-)). \quad (2.19)$$

Then, by [K12, Theorem 3.10], it holds

$$\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(W)|_{\mathfrak{g}'} \simeq \bigoplus_{V'} m(W, V') \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(V') \quad (2.20)$$

in the Grothendieck group $K(\mathcal{O}^{\mathfrak{p}'})$ of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}'}$. The isomorphism (2.20) is equivalent to the equality of formal characters of both sides.

For the convenience of the reader, we recall the main arguments of the proof of (2.20). First, the identifications $\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(W) \simeq U(\mathfrak{n}_-) \otimes W \simeq S(\mathfrak{n}_-) \otimes W$ imply that on MA^+ the formal character of this module is given by

$$\begin{aligned} \text{ch}(\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(W))(ma) &= \sum_{N \geq 0} \text{tr}(\text{Ad}(ma)|_{S^N(\mathfrak{n}_-)}) \text{ch}(W)(ma) \\ &= \det(1 - \text{Ad}(ma)|_{\mathfrak{n}_-})^{-1} \text{ch}(W)(ma), \quad ma \in MA^+ \subset L. \end{aligned}$$

We restrict this formula to $M'A^+ \subset L'$. The relation

$$\begin{aligned} \det(1 - \text{Ad}(m'a)|_{\mathfrak{n}_-})^{-1} &= \det(1 - \text{Ad}(m'a)|_{\mathfrak{n}'_-})^{-1} \det(1 - \text{Ad}(a)|_{\mathfrak{n}_-/\mathfrak{n}'_-})^{-1} \\ &= \det(1 - \text{Ad}(m'a)|_{\mathfrak{n}'_-})^{-1} \sum_{N \geq 0} \text{tr}(\text{Ad}(a)|_{S^N(\mathfrak{n}_-/\mathfrak{n}'_-)}) \end{aligned} \quad (2.21)$$

for $m'a \in M'A^+$ yields

$$\begin{aligned} \text{ch}(\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(W))(m'a) &= \det(1 - \text{Ad}(m'a)|_{\mathfrak{n}'_-})^{-1} \text{ch}(W \otimes S(\mathfrak{n}_-/\mathfrak{n}'_-))(m'a) \\ &= \det(1 - \text{Ad}(m'a)|_{\mathfrak{n}'_-})^{-1} \sum_{V'} m(W, V') \text{ch}(V')(m'a) \\ &= \sum_{V'} m(W, V') \text{ch}(\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(V'))(m'a). \end{aligned}$$

This proves the assertion.⁷

The isomorphism (2.20) may be regarded as the main step in the determination of a branching rule of the restriction to \mathfrak{g}' of the generalized Verma modules $\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(W)$. We also stress that the modules on the right-hand side of (2.20) may be reducible: this effect will actually play a role later.

Now let $W = V_{p,\lambda}$. In order to determine the multiplicities

$$m_{V'}(p, \lambda) \stackrel{\text{def}}{=} m(V_{p,\lambda}, V'), \quad (2.22)$$

we distinguish several cases.

The generic case. We assume that n is odd and $p \neq \frac{n+1}{2}$ or n is even and $p \neq \frac{n}{2}$. Then Lemma 2.2.2 implies that the $SO(n, \mathbb{R})$ -module $\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}$ is irreducible and the multiplicity (2.22) equals one iff

$$V' \simeq \Lambda^p(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N} \quad \text{or} \quad V' \simeq \Lambda^{p-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}$$

⁷The identity (2.21) also implies an identity for Selberg zeta functions which in turn suggests part of the theory in [J09] (as explained in Section 1.2 of this reference).

for some $N \in \mathbb{N}_0$.

Middle degree cases (n odd). If $p = \frac{n-1}{2}$, the multiplicity (2.22) equals one iff

$$V' \simeq \Lambda_+^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N} \text{ or } V' \simeq \Lambda_-^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N} \text{ or } V' \simeq \Lambda_-^{\frac{n-3}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}$$

for some $N \in \mathbb{N}_0$. Similarly, if $p = \frac{n+1}{2}$, the multiplicity (2.22) equals one iff

$$V' \simeq \Lambda_-^{\frac{n+1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N} \text{ or } V' \simeq \Lambda_+^{\frac{n+1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N} \text{ or } V' \simeq \Lambda_-^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}$$

for some $N \in \mathbb{N}_0$.

Middle degree cases (n even). Assume that $p = \frac{n}{2}$. Then Lemma 2.2.2 shows that the multiplicity (2.22) is one iff

$$V' \simeq \Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N} \simeq \Lambda^{\frac{n}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}$$

for some $N \in \mathbb{N}_0$.

Using these observations, the following results are special cases of (2.20).

Proposition 2.3.1 (Character identities. Generic cases). *We consider the compatible pair of simple Lie algebras*

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{so}(n+1, 1, \mathbb{R}), \quad \mathfrak{g}'(\mathbb{R}) = \mathfrak{so}(n, 1, \mathbb{R})$$

and their respective conformal parabolic subalgebras $\mathfrak{p}(\mathbb{R})$ and $\mathfrak{p}'(\mathbb{R})$. Assume that $p \neq \frac{n+1}{2}$ if n is odd and $p \neq \frac{n}{2}$ if n is even. Then, in the Grothendieck group $K(\mathcal{O}^{\mathfrak{p}})$ of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$, it holds

$$\begin{aligned} \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}_{\lambda})|_{\mathfrak{g}'} \\ \simeq \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^{p-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \oplus \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^p(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}). \end{aligned} \quad (2.23)$$

For generic λ , the identity (2.23) refines to an actual branching law (with direct sums of irreducible modules on the right hand-side).

Proposition 2.3.2 (Character identities. Middle degree cases). *With the same notation as in Proposition 2.3.1, it holds*

$$\begin{aligned} \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(\Lambda^{\frac{n-1}{2}}(\mathbb{R}^n) \otimes \mathbb{C}_{\lambda})|_{\mathfrak{g}'} \\ \simeq \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda_+^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \oplus \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda_-^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \\ \oplus \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^{\frac{n-3}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \end{aligned} \quad (2.24)$$

and

$$\begin{aligned} \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(\Lambda^{\frac{n+1}{2}}(\mathbb{R}^n) \otimes \mathbb{C}_{\lambda})|_{\mathfrak{g}'} \\ \simeq \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^{\frac{n+1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \\ \oplus \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda_+^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \oplus \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda_-^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \end{aligned} \quad (2.25)$$

for odd n and

$$\mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}\left(\Lambda_{\pm}^{\frac{n}{2}}(\mathbb{R}^n) \otimes \mathbb{C}_{\lambda}\right)|_{\mathfrak{g}'} \simeq \bigoplus_{N \in \mathbb{N}_0} \mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}\left(\Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-N}\right) \quad (2.26)$$

for even n .

The F -method is a procedure to construct the emdeddings of the $\mathcal{U}(\mathfrak{g}')$ -submodules in the decompositions in Proposition 2.3.1 and Proposition 2.3.2. It has been explained in Section 2.1. Its realization requires to work with the induced representations in the non-compact model. The following lemma provides the necessary details. In the non-compact model of the infinitesimal induced representation $d\pi_{\lambda,p}$, the $\mathfrak{g}(\mathbb{R})$ -module $\text{Ind}_P^G(V_{\lambda,p})$ is given by the space $C^\infty(\mathfrak{n}_-(\mathbb{R})) \otimes V_{\lambda,p} \simeq C^\infty(\mathbb{R}^n) \otimes V_{\lambda,p}$ of smooth $V_{\lambda,p}$ -valued functions on $\mathfrak{n}_-(\mathbb{R}) \simeq \mathbb{R}^n$. We shall write $\mathfrak{n}_-(\mathbb{R}) \ni X = \sum_{k=1}^n x_k E_j^-$ and $\mathfrak{n}_-^*(\mathbb{R}) \ni \xi = \sum_{k=1}^n \xi_k (E_k^-)^*$.

Lemma 2.3.3. (1) The operator $d\pi_{\lambda,p}(E_j^+)$, $j = 1, \dots, n-1$, acts on $C^\infty(\mathbb{R}^n) \otimes V_{\lambda,p}$ by

$$\begin{aligned} d\pi_{\lambda,p}(E_j^+)(u \otimes \omega)(x) = & \left(-\frac{1}{2} \sum_{k=1}^n x_k^2 \partial_{x_j} + x_j \left(\lambda + \sum_{k=1}^n x_k \partial_{x_k} \right) \right) (u) \otimes \omega \\ & - \sum_{k=1}^n x_k u \otimes (E_k^- \otimes E_j^+ - E_j^- \otimes E_k^+)(\omega), \end{aligned} \quad (2.27)$$

where $u \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Lambda^p(\mathbb{R}^n)$.

(2) The operator $d\tilde{\pi}_{\lambda,p}(E_j^+)$, $j = 1, \dots, n-1$, acts on $\text{Pol}(\mathbb{R}^n) \otimes V_{\lambda,p}$ by

$$\begin{aligned} d\tilde{\pi}_{\lambda,p}(E_j^+)(p \otimes \omega) = & -i \left(\frac{1}{2} \xi_j \Delta_\xi + (\lambda - E_\xi) \partial_{\xi_j} \right) (p) \otimes \omega \\ & + i \sum_{k=1}^n \partial_{\xi_k} (p) \otimes (E_k^- \otimes E_j^+ - E_j^- \otimes E_k^+)(\omega), \end{aligned} \quad (2.28)$$

where $p \in \text{Pol}(\mathbb{R}^n)$ and $\omega \in \Lambda^p(\mathbb{R}^n)$. Here $i \in \mathbb{C}$ is the complex unit, $\Delta_\xi = \sum_{k=1}^n \partial_{\xi_k}^2$ is the Laplace operator in the variables ξ_1, \dots, ξ_n of $\mathfrak{n}_-^*(\mathbb{R})$ and $E_\xi = \sum_{k=1}^n \xi_k \partial_{\xi_k}$ is the Euler operator.

(3) The operator $(d\pi_{\lambda,p})^\vee(E_j^+)$, $j = 1, \dots, n-1$, acts on $C^\infty(\mathbb{R}^n) \otimes V_{\lambda,p}^\vee$ by

$$\begin{aligned} (d\pi_{\lambda,p})^\vee(E_j^+)(u \otimes \omega)(x) = & \left(-\frac{1}{2} \sum_{k=1}^n x_k^2 \partial_{x_j} + x_j \left(-\lambda + \sum_{k=1}^n x_k \partial_{x_k} \right) \right) (u) \otimes \omega \\ & + \sum_{k=1}^n x_k u \otimes ((E_j^+)^* \otimes (E_k^-)^* - (E_k^+)^* \otimes (E_j^-)^*)(\omega), \end{aligned} \quad (2.29)$$

where $u \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Lambda^p(\mathbb{R}^n)^*$.

We recall that through the identifications $\mathfrak{n}_\pm(\mathbb{R}) \simeq \mathbb{R}^n$ the elements $E_k^- \otimes E_j^+ - E_j^- \otimes E_k^+$ are regarded as elements of $\mathfrak{so}(n, \mathbb{R})$. They act on $\Lambda^p(\mathbb{R}^n)$ as stated in Remark 2.2.1.

We also note that the formulation of Lemma 2.3.3 has been simplified by suppressing the tensor products with \mathbb{C}_λ .

Proof. (i) The proof is a minor extension of the proof of [KOSS15, Lemma 4.1]. It basically suffices to determine the additional contribution by the action on $\Lambda^p(\mathbb{R}^n)$. We calculate

in the matrix realization. Let \cdot denote the matrix product. The result of the left action of $n = \exp(Z) \in N_+$ on $x = \exp(X) \in N_-$ is

$$n^{-1} \cdot x = \begin{pmatrix} a & -Z + \frac{1}{2}|Z|^2 X^t & -\frac{1}{2}|Z|^2 \\ X - \frac{1}{2}|X|^2 Z^t & \text{Id} - Z^t \otimes X^t & Z^t \\ -\frac{1}{2}|X|^2 & -X^t & 1 \end{pmatrix},$$

where $a \stackrel{\text{def}}{=} 1 - Z \cdot X + \frac{|Z|^2|X|^2}{4}$. Here we identify Z with an arrow vector and X with a column vector. If Z and X are sufficiently small, we have $a \neq 0$, and this element decomposes as a product $\tilde{x} \cdot p$, where $\tilde{x} \in N_-$ and

$$p = \begin{pmatrix} a & * & * \\ 0 & m & * \\ 0 & 0 & a^{-1} \end{pmatrix} \in P.$$

The elements a , m and \tilde{x} are given up to the first-order terms in $|Z|$ by

$$a \sim 1 - Z \cdot X \quad \text{and} \quad m \sim \text{Id} - Z^t \otimes X^t + X \otimes Z \quad (2.30)$$

and

$$\tilde{x} = \exp(\tilde{X}), \quad \tilde{X} = (1 + Z \cdot X)(X - \frac{1}{2}|X|^2 Z^t). \quad (2.31)$$

Now let $X = \sum_{k=1}^n x_k E_k^- \in \mathfrak{n}_-$, $Z = tE_j^+ \in \mathfrak{n}_+$, and let $a_j(t)$, $m_j(t)$ and $\tilde{x}_j(t)$ be the corresponding 1-parameter subgroups of elements in (2.30) and (2.31). Then using $(E_j^+)^t = E_j^-$ we find

$$d\xi_\lambda \left(\frac{d}{dt} \Big|_{t=0} (a_j(t)) \right) = -\lambda x_j$$

and

$$\frac{d}{dt} \Big|_{t=0} (m_j(t)) = \sum_{k=1}^n x_k (E_k^- \otimes E_j^+ - E_j^- \otimes E_k^+) \in \mathfrak{so}(n, \mathbb{R}). \quad (2.32)$$

Now, for $u \in \text{Ind}_P^G(V_{\lambda,p})$, we obtain

$$\pi_{\lambda,p}(\exp(tE_j^+))(u)(x) = u(\exp(-tE_j^+) \cdot x) = \xi_\lambda(a_j(t)^{-1})\sigma_p(m_j(t)^{-1})u(\tilde{x}_j(t)).$$

The assertion (i) follows by differentiation.

(ii) The representation $d\tilde{\pi}_{\lambda,p}$ is defined by Fourier transform of the non-compact model of the induced representation with the inducing module $V_{\lambda,p}$. Therefore, the formula for the action of $d\tilde{\pi}_{\lambda,p}(E_j^+)$ follows from the formula in (i) in two steps: first reverse the order of the composition and then apply the Fourier transform using $x_j \mapsto -i\partial_{\xi_j}$ and $\partial_{x_j} \mapsto -i\xi_j$ preserving the order of the composition.

(iii) We recall that $\mathbb{C}_{-\lambda}$ is the dual of \mathbb{C}_λ . Moreover, the dual of the action of $E_k^- \otimes E_j^+$ on $\Lambda^p(\mathbb{R}^n)$ is given by the negative of the action of $(E_j^+)^* \otimes (E_k^-)^*$ on $\Lambda^p(\mathbb{R}^n)^*$. Hence the assertion follows from (i). \square

3. SINGULAR VECTORS

In this section, we determine the singular vectors

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^q(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \text{Pol}_N(\mathfrak{n}_-^*(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

which correspond to the homomorphisms

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^q(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

of generalized Verma modules. We apply the F -method and proceed as follows. We first determine the consequences of the \mathfrak{l}' -equivariance of (the Fourier transform of) singular vectors. This leads to a natural distinction between four types of singular vectors together with a natural ansatz in each case. The remaining analysis of their \mathfrak{n}'_+ -equivariance will be carried out in four separate sections according to the type. These results are the key technical results of the paper.

3.1. The \mathfrak{l}' -equivariance. We study homomorphisms of the form

$$\mathrm{Hom}_{SO(n-1)}(\Lambda^q(\mathbb{R}^{n-1})^* \otimes \mathbb{C}, \mathrm{Pol}_N(\mathbb{R}^n) \otimes \Lambda^p(\mathbb{R}^n)^*) \quad (3.1)$$

for arbitrary p and q and all $N \in \mathbb{N}_0$. Here $\mathrm{Pol}_N(\mathbb{R}^n)$ denotes the vector space of complex-valued polynomials on \mathbb{R}^n which are homogenous of degree $N \in \mathbb{N}_0$. The results will be applied to evaluate the \mathfrak{l}' -equivariance of singular vectors and provide a natural ansatz for the analysis of their \mathfrak{n}'_+ -invariance.

Among the homomorphisms of the form (3.1), those in the spaces

$$\mathrm{Hom}_{SO(n-1)}(\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C}, \mathrm{Pol}_N(\mathbb{R}^n) \otimes \Lambda^p(\mathbb{R}^n)^*) \quad (3.2)$$

and

$$\mathrm{Hom}_{SO(n-1)}(\Lambda^{p-1}(\mathbb{R}^{n-1})^* \otimes \mathbb{C}, \mathrm{Pol}_N(\mathbb{R}^n) \otimes \Lambda^p(\mathbb{R}^n)^*) \quad (3.3)$$

will play a dominant role in what follows.

We identify the spaces

$$\mathrm{Pol}_N^p(\mathbb{R}^n) \stackrel{\mathrm{def}}{=} \mathrm{Pol}_N(\mathbb{R}^n) \otimes \Lambda^p(\mathbb{R}^n)^*$$

with the respective subspaces of $\Omega^p(\mathbb{R}^n)$ consisting of differential p -forms on \mathbb{R}^n with complex-valued homogeneous polynomial coefficients of degree N . In particular, we apply to the elements of $\mathrm{Pol}_N^p(\mathbb{R}^n)$ the usual rules of the calculus of differential forms.

The spaces $\Lambda^p(\mathbb{R}^n)$, $\Lambda^p(\mathbb{R}^n)$ and $\mathrm{Pol}_N(\mathbb{R}^n)$ will be regarded as $SO(n, \mathbb{R})$ -modules with the usual induced and push-forward actions, respectively. The fact that some of the modules $\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}$ are reducible (Lemma 2.2.2) will play only a minor role in the following and henceforth will be suppressed for the sake of uniform statements.

We use Euclidean coordinates x_1, \dots, x_n on \mathbb{R}^n , basis tangential vectors $\partial_1, \dots, \partial_n$ and dual covectors dx_1, \dots, dx_n . Let d and δ denote the exterior differential and co-differential on differential forms on \mathbb{R}^n , respectively. Let $\Delta = \delta d + d\delta$ be the corresponding Laplacian on forms. Let x_n be the normal variable for the subspace \mathbb{R}^{n-1} .⁸

Any p -form $\omega \in \mathrm{Pol}_N^p(\mathbb{R}^n)$, $p \geq 1$, admits a normal Taylor series

$$\omega = \sum_{j=0}^N x_n^{N-j} (\omega_j^+ + dx_n \wedge \omega_j^-). \quad (3.4)$$

Hence any

$$\omega \in \mathrm{Hom}_{SO(n-1)}(\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C}, \mathrm{Pol}_N^p(\mathbb{R}^n)), \quad p \geq 1$$

has Taylor coefficients

$$\omega_j^+ \in \mathrm{Hom}_{SO(n-1)}(\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C}, \mathrm{Pol}_j^p(\mathbb{R}^{n-1}))$$

and

$$\omega_j^- \in \mathrm{Hom}_{SO(n-1)}(\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C}, \mathrm{Pol}_j^{p-1}(\mathbb{R}^{n-1}));$$

⁸In later sections, we shall use the notation \bar{d} , $\bar{\delta}$ and $\bar{\Delta}$ for these operators.

for $p = 0$, the components ω_j^- vanish, of course.

Next, we recall the decomposition

$$\text{Pol}_N(\mathbb{R}^n) \simeq \bigoplus_{k=0}^{\lfloor \frac{N}{2} \rfloor} r^{2k} \mathcal{H}_{N-2k}(\mathbb{R}^n), \quad r^2 = |x|^2,$$

where the space $\mathcal{H}_N(\mathbb{R}^n) = \ker \Delta|_{\text{Pol}_N(\mathbb{R}^n)}$ of homogeneous harmonic polynomials of degree N is an irreducible $SO(n, \mathbb{R})$ -module of highest weight $N\Lambda_1$.

The following result extends this fact to polynomial forms. Let

$$\mathcal{H}_N^p(\mathbb{R}^n) = \{\omega \in \text{Pol}_N^p(\mathbb{R}^n) \mid \Delta\omega = 0, \delta\omega = 0\} \subset \Omega^p(\mathbb{R}^n)$$

be the subspace of co-closed *harmonic* homogeneous polynomial p -forms ω of degree N . The element $\alpha \stackrel{\text{def}}{=} \sum_{j=1}^n x_j \otimes dx_j \in \text{Pol}_1^1(\mathbb{R}^n)$ defines a map

$$\alpha \wedge : \text{Pol}_N^p(\mathbb{R}^n) \rightarrow \text{Pol}_{N+1}^{p+1}(\mathbb{R}^n)$$

by $\alpha \wedge (p \otimes dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_p}) = \sum_{j=1}^n x_j p \otimes dx_j \wedge dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_p}$. Furthermore, the element $E \stackrel{\text{def}}{=} \sum_{j=1}^n x_j \otimes \partial_j$ defines a map

$$i_E : \text{Pol}_N^p(\mathbb{R}^n) \rightarrow \text{Pol}_{N+1}^{p-1}(\mathbb{R}^n)$$

by $i_E(p \otimes \omega) = \sum_{j=1}^n x_j p \otimes i_{\partial_j}(\omega)$.

Proposition 3.1.1 ([IT78], Theorem 6.8). *(1) For $p = 1, \dots, n$ and $N \in \mathbb{N}_0$, we have the decomposition*

$$\text{Pol}_N^p(\mathbb{R}^n) = \mathcal{H}_N^p(\mathbb{R}^n) \oplus (r^2 \text{Pol}_{N-2}^p(\mathbb{R}^n) + \alpha \wedge \text{Pol}_{N-1}^{p-1}(\mathbb{R}^n)), \quad r^2 = |x|^2.$$

Moreover, the $SO(n, \mathbb{R})$ -module $\mathcal{H}_N^p(\mathbb{R}^n)$ decomposes as

$$\mathcal{H}_N^p(\mathbb{R}^n) = {}'\mathcal{H}_N^p(\mathbb{R}^n) \oplus {}''\mathcal{H}_N^p(\mathbb{R}^n), \quad p+N \neq 0, \quad n-p+N \neq 0,$$

where

$${}'\mathcal{H}_N^p(\mathbb{R}^n) = \mathcal{H}_N^p(\mathbb{R}^n) \cap \ker d \quad \text{and} \quad {}''\mathcal{H}_N^p(\mathbb{R}^n) = \mathcal{H}_N^p(\mathbb{R}^n) \cap \ker i_E.$$

(2) For odd n , the $SO(n, \mathbb{R})$ -module ${}'\mathcal{H}_N^p(\mathbb{R}^n)$ is irreducible and has highest weights

$$\begin{cases} N\Lambda_1 + \Lambda_p, & 0 < p < \frac{n-1}{2} \\ N\Lambda_1 + 2\Lambda_{\frac{n-1}{2}}, & p = \frac{n-1}{2}, \frac{n+1}{2} \\ N\Lambda_1 + \Lambda_{n-p}, & \frac{n+1}{2} < p \leq n-1 \end{cases}$$

or, equivalently,

$$\begin{cases} N1_1 + 1_p & 0 < p \leq \frac{n-1}{2} \\ N1_1 + 1_{n-p}, & \frac{n+1}{2} \leq p \leq n-1. \end{cases}$$

(3) For odd n , the $SO(n, \mathbb{R})$ -module ${}''\mathcal{H}_N^p(\mathbb{R}^n)$ ($N \geq 1$) is irreducible and has highest weights

$$\begin{cases} (N-1)\Lambda_1 + \Lambda_{p+1}, & 0 \leq p < \frac{n-3}{2} \\ (N-1)\Lambda_1 + 2\Lambda_{\frac{n-1}{2}}, & p = \frac{n-3}{2}, \frac{n-1}{2} \\ (N-1)\Lambda_1 + \Lambda_{n-p-1}, & \frac{n+1}{2} \leq p \leq n-1 \end{cases}$$

or, equivalently,

$$\begin{cases} (N-1)1_1 + 1_{p+1}, & 0 \leq p \leq \frac{n-3}{2} \\ (N-1)1_1 + 1_{n-p-1}, & \frac{n-1}{2} \leq p \leq n-1. \end{cases}$$

(4) For even n , the $SO(n, \mathbb{R})$ -module $'\mathcal{H}_N^p(\mathbb{R}^n)$ decomposes into irreducible modules with highest weights

$$\begin{cases} N\Lambda_1 + \Lambda_p, & 0 < p < \frac{n}{2} - 1 \\ N\Lambda_1 + \Lambda_{\frac{n}{2}-1} + \Lambda_{\frac{n}{2}}, & p = \frac{n}{2} - 1, \frac{n}{2} + 1 \\ N\Lambda_1 + 2\Lambda_{\frac{n}{2}-1}, N\Lambda_1 + 2\Lambda_{\frac{n}{2}}, & p = \frac{n}{2} \\ N\Lambda_1 + \Lambda_{n-p}, & \frac{n}{2} + 1 < p \leq n-1 \end{cases}$$

or, equivalently,

$$\begin{cases} N1_1 + 1_p, & 0 < p \leq \frac{n}{2} - 1 \\ N1_1 + 1_{\frac{n}{2}}^+, N1_1 + 1_{\frac{n}{2}}^-, & p = \frac{n}{2} \\ N1_1 + 1_{n-p}, & \frac{n}{2} + 1 \leq p \leq n-1. \end{cases}$$

The decompositions are multiplicity free.

(5) For even n , the $SO(n, \mathbb{R})$ -module $''\mathcal{H}_N^p(\mathbb{R}^n)$ ($N \geq 1$) decomposes into irreducible modules with highest weights

$$\begin{cases} (N-1)\Lambda_1 + \Lambda_{p+1}, & 0 \leq p < \frac{n}{2} - 2 \\ (N-1)\Lambda_1 + \Lambda_{\frac{n}{2}-1} + \Lambda_{\frac{n}{2}}, & p = \frac{n}{2} - 2, \frac{n}{2} \\ (N-1)\Lambda_1 + 2\Lambda_{\frac{n}{2}-1}, (N-1)\Lambda_1 + 2\Lambda_{\frac{n}{2}}, & p = \frac{n}{2} - 1 \\ (N-1)\Lambda_1 + \Lambda_{n-p-1}, & \frac{n}{2} < p \leq n-1 \end{cases}$$

or, equivalently,

$$\begin{cases} (N-1)1_1 + 1_{p+1}, & 0 \leq p \leq \frac{n}{2} - 2 \\ (N-1)1_1 + 1_{\frac{n}{2}}^+, (N-1)1_1 + 1_{\frac{n}{2}}^-, & p = \frac{n}{2} - 1 \\ (N-1)1_1 + 1_{n-p-1}, & \frac{n}{2} \leq p \leq n-1. \end{cases}$$

The decompositions are multiplicity free.

(6) We have

$$\mathcal{H}_0^0(\mathbb{R}^n) = '\mathcal{H}_0^0(\mathbb{R}^n) = ''\mathcal{H}_0^0(\mathbb{R}^n) = \mathbb{C} \quad \text{and} \quad \mathcal{H}_0^n(\mathbb{R}^n) = '\mathcal{H}_0^n(\mathbb{R}^n) = \mathbb{C}. \quad (3.5)$$

Moreover, $'\mathcal{H}_N^0(\mathbb{R}^n) = 0$ for $N \geq 1$ and $''\mathcal{H}_0^p(\mathbb{R}^n) = 0$ for $0 < p \leq n$.

Proposition 3.1.1/(1) implies that any element of $\text{Pol}_N^p(\mathbb{R}^{n-1})$ can be decomposed according to

$$\mathcal{H}_N^p(\mathbb{R}^{n-1}) \oplus r^2 \mathcal{H}_{N-2}^p(\mathbb{R}^{n-1}) \oplus \cdots + \alpha \wedge (\mathcal{H}_{N-1}^{p-1}(\mathbb{R}^{n-1}) \oplus r^2 \mathcal{H}_{N-3}^{p-1}(\mathbb{R}^{n-1}) \oplus \cdots), \quad (3.6)$$

where $r^2 = |x'|^2 = \sum_{k=1}^{n-1} x_k^2$ and $\alpha = \sum_{i=1}^{n-1} x_i \otimes dx_i$, $x' \in \mathbb{R}^{n-1}$.

In the following, spaces of harmonic polynomials on \mathbb{R}^{n-1} will be denoted by \mathcal{H}_N^p , $'\mathcal{H}_N^p$ and $''\mathcal{H}_N^p$. We shall also assume that $p \neq \frac{n-1}{2}$ if n is odd. Then the $SO(n-1, \mathbb{R})$ -module $\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C}$ is irreducible and has highest weight 1_p or 1_{n-1-p} depending on the size of p .

We continue with the discussion of the structure of the spaces (3.2). For this purpose, we need to find the contributions of the irreducible $SO(n-1, \mathbb{R})$ -module of respective

highest weights 1_p and 1_{n-1-p} to $\text{Pol}_N^p(\mathbb{R}^n)$. For this purpose, we apply the decomposition (3.6) to the Taylor coefficients ω_j^\pm of $\omega \in \text{Pol}_N^p(\mathbb{R}^n)$ in (3.4).

We start with the discussion of the terms ω_j^+ .

First, by Proposition 3.1.1/(1), the decomposition (3.6) of the space $\text{Pol}_{2j+1}^p(\mathbb{R}^{n-1})$ of polynomial forms of *odd degree* on \mathbb{R}^{n-1} contains the $SO(n-1, \mathbb{R})$ -modules

$$' \mathcal{H}_{2k+1}^p, '' \mathcal{H}_{2k+1}^p, \quad 0 \leq k \leq j \quad (3.7)$$

and

$$\alpha \wedge ' \mathcal{H}_{2k}^{p-1}, \alpha \wedge '' \mathcal{H}_{2k}^{p-1}, \quad 0 \leq k \leq j. \quad (3.8)$$

Now assume that n is even. By Proposition 3.1.1/(2),(3), these modules are of respective highest weights

$$\begin{cases} (2k+1)1_1 + 1_p, & 0 < p \leq \frac{n}{2} - 1 \\ (2k+1)1_1 + 1_{n-1-p}, & \frac{n}{2} \leq p \leq n-2 \end{cases}, \quad \begin{cases} (2k)1_1 + 1_{p+1}, & 0 \leq p \leq \frac{n}{2} - 2 \\ (2k)1_1 + 1_{n-2-p}, & \frac{n}{2} - 1 \leq p \leq n-2 \end{cases}$$

and

$$\begin{cases} (2k)1_1 + 1_{p-1}, & 1 < p \leq \frac{n}{2} \\ (2k)1_1 + 1_{n-p}, & \frac{n}{2} + 1 \leq p \leq n-1 \end{cases}, \quad \begin{cases} (2k-1)1_1 + 1_p, & 1 \leq p \leq \frac{n}{2} - 1 \\ (2k-1)1_1 + 1_{n-1-p}, & \frac{n}{2} \leq p \leq n-1. \end{cases}$$

Among these contributions, the module of highest weight 1_p (for $0 \leq p \leq \frac{n}{2} - 1$) appears iff $p = \frac{n}{2} - 1$ and it is realized as

$$r^{2j} '' \mathcal{H}_1^{\frac{n}{2}-1} \subset \text{Pol}_{2j+1}^{\frac{n}{2}-1}(\mathbb{R}^{n-1}).$$

The module $'' \mathcal{H}_1^{\frac{n}{2}-1}$ is isomorphic to $\mathcal{H}_0^{\frac{n}{2}}$ embedded into $\text{Pol}_1^{\frac{n}{2}-1}(\mathbb{R}^{n-1})$ by $\omega \mapsto i_E(\omega)$. Hence it coincides with the range of the map

$$\Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto i_E(\star \omega) \in \text{Pol}_1^{\frac{n}{2}-1}(\mathbb{R}^{n-1}).$$

Thus, for even n , we find the homomorphisms

$$\Lambda^{\frac{n}{2}-1}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j-1} r^{2j} i_E(\star \omega) \in \text{Pol}_N^{\frac{n}{2}-1}(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N-1 \quad (3.9)$$

in the spaces (3.2).

Similarly, the module of highest weight 1_{n-1-p} (for $\frac{n}{2} \leq p \leq n-1$) appears iff $p = \frac{n}{2}$ and it is realized as

$$r^{2j} \alpha \wedge ' \mathcal{H}_0^{\frac{n}{2}-1} \subset \text{Pol}_{2j+1}^{\frac{n}{2}}(\mathbb{R}^{n-1}).$$

Thus, for even n , we find the homomorphisms

$$\Lambda^{\frac{n}{2}}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j-1} r^{2j} \alpha \wedge \star \omega \in \text{Pol}_N^{\frac{n}{2}}(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N-1 \quad (3.10)$$

in the spaces (3.2).

Now assume that n is odd. By Proposition 3.1.1/(4),(5), the modules (3.7) and (3.8) are of respective highest weights

$$\begin{cases} (2k+1)1_1 + 1_p, & 0 < p \leq \frac{n-3}{2} \\ (2k+1)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n-1}{2} \\ (2k+1)1_1 + 1_{n-1-p}, & \frac{n+1}{2} \leq p \leq n-2 \end{cases}, \quad \begin{cases} (2k)1_1 + 1_{p+1}, & 0 \leq p \leq \frac{n-5}{2} \\ (2k)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n-3}{2} \\ (2k)1_1 + 1_{n-2-p}, & \frac{n-1}{2} \leq p \leq n-2 \end{cases}$$

and

$$\begin{cases} (2k)1_1 + 1_{p-1}, & 1 < p \leq \frac{n-1}{2} \\ (2k)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n+1}{2} \\ (2k)1_1 + 1_{n-p}, & \frac{n+3}{2} \leq p \leq n-1 \end{cases}, \quad \begin{cases} (2k-1)1_1 + 1_p, & 1 \leq p \leq \frac{n-3}{2} \\ (2k-1)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n-1}{2} \\ (2k-1)1_1 + 1_{n-1-p}, & \frac{n+1}{2} \leq p \leq n-1. \end{cases}$$

The modules of highest weights 1_p and 1_{n-1-p} do not appear among these modules.

Second, we consider the space $\text{Pol}_{2j}^p(\mathbb{R}^{n-1})$ of polynomial forms of *even degree* $2j$. The corresponding decomposition (3.6) contains the $SO(n-1, \mathbb{R})$ -modules

$$' \mathcal{H}_{2k}^p, '' \mathcal{H}_{2k}^p, \quad 0 \leq k \leq j \quad (3.11)$$

and

$$\alpha \wedge ' \mathcal{H}_{2k-1}^{p-1}, \alpha \wedge '' \mathcal{H}_{2k-1}^{p-1}, \quad 0 \leq k \leq j. \quad (3.12)$$

Now assume that n is even. By Proposition 3.1.1/(2),(3), these modules are of respective highest weights

$$\begin{cases} (2k)1_1 + 1_p, & 0 < p \leq \frac{n}{2} - 1 \\ (2k)1_1 + 1_{n-1-p}, & \frac{n}{2} \leq p \leq n-2 \end{cases}, \quad \begin{cases} (2k-1)1_1 + 1_{p+1}, & 0 \leq p \leq \frac{n}{2} - 2 \\ (2k-1)1_1 + 1_{n-2-p}, & \frac{n}{2} - 1 \leq p \leq n-2 \end{cases}$$

and

$$\begin{cases} (2k-1)1_1 + 1_{p-1}, & 1 < p \leq \frac{n}{2} \\ (2k-1)1_1 + 1_{n-p}, & \frac{n}{2} + 1 \leq p \leq n-1 \end{cases}, \quad \begin{cases} (2k-2)1_1 + 1_p, & 1 \leq p \leq \frac{n}{2} - 1 \\ (2k-2)1_1 + 1_{n-1-p}, & \frac{n}{2} \leq p \leq n-1. \end{cases}$$

It follows that the representations of highest weights 1_p and 1_{n-1-p} are realized by

$$r^{2j'} \mathcal{H}_0^p \quad \text{and} \quad r^{2j-2} \alpha \wedge '' \mathcal{H}_1^{p-1}.$$

The module $'' \mathcal{H}_1^{p-1}$ is isomorphic to \mathcal{H}_0^p embedded into $\text{Pol}_1^{p-1}(\mathbb{R}^{n-1})$ by $\omega \mapsto i_E(\omega)$. In fact, it is isomorphic to $\ker i_E = i_E(\mathcal{H}_0^p)$. Thus, we find the homomorphisms

$$\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \simeq \mathcal{H}_0^p \ni \omega \mapsto x_n^{N-2j} r^{2j} \omega \in \text{Pol}_N^p(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N \quad (3.13)$$

and

$$\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \simeq \mathcal{H}_0^p \ni \omega \mapsto x_n^{N-2j} r^{2j-2} \alpha \wedge i_E(\omega) \in \text{Pol}_N^p(\mathbb{R}^{n-1}), \quad 2 \leq 2j \leq N \quad (3.14)$$

in the spaces (3.2).

Now assume that n is odd. By Proposition 3.1.1/(4),(5), the modules (3.11) and (3.12) are of respective highest weights

$$\begin{cases} (2k)1_1 + 1_p, & 0 < p \leq \frac{n-3}{2} \\ (2k)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n-1}{2} \\ (2k)1_1 + 1_{n-1-p}, & \frac{n+1}{2} \leq p \leq n-2 \end{cases}, \quad \begin{cases} (2k-1)1_1 + 1_{p+1}, & 0 \leq p \leq \frac{n-5}{2} \\ (2k-1)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n-3}{2} \\ (2k-1)1_1 + 1_{n-2-p}, & \frac{n-1}{2} \leq p \leq n-2 \end{cases}$$

and

$$\begin{cases} (2k-1)1_1 + 1_{p-1}, & 1 < p \leq \frac{n-1}{2} \\ (2k-1)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n+1}{2} \\ (2k-1)1_1 + 1_{n-p}, & \frac{n+3}{2} \leq p \leq n-1 \end{cases}, \quad \begin{cases} (2k-2)1_1 + 1_p, & 1 \leq p \leq \frac{n-3}{2} \\ (2k-2)1_1 + 1_{\frac{n-1}{2}}, & p = \frac{n-1}{2} \\ (2k-2)1_1 + 1_{n-1-p}, & \frac{n+1}{2} \leq p \leq n-1. \end{cases}$$

Again, it follows that the representations of highest weights 1_p and 1_{n-1-p} are realized by

$$r^{2j'} \mathcal{H}_0^p \quad \text{and} \quad r^{2j-2} \alpha \wedge '' \mathcal{H}_1^{p-1}.$$

This yields the homomorphisms (3.13) and (3.14). Moreover, in this case, we find the additional homomorphisms

$$\Lambda^{\frac{n-1}{2}}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j} r^{2j} \star \omega \in \text{Pol}_{\frac{n-1}{2}}(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N \quad (3.15)$$

and

$$\Lambda^{\frac{n-1}{2}}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j} r^{2j-2} \alpha \wedge i_E(\star \omega) \in \text{Pol}_{\frac{n-1}{2}}(\mathbb{R}^{n-1}), \quad 2 \leq 2j \leq N. \quad (3.16)$$

This finishes the discussion of the contributions of the terms ω_j^+ .

We continue with a summary of an analogous discussion of the contributions by the terms ω_j^- . Similar arguments as above imply that the terms ω_{2j}^- of *even degree* only contribute the homomorphisms

$$\Lambda^{\frac{n}{2}}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j} r^{2j} dx_n \wedge \star \omega \in \text{Pol}_{\frac{n}{2}}(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N \quad (3.17)$$

and

$$\Lambda^{\frac{n}{2}}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j} r^{2j} dx_n \wedge \alpha \wedge i_E(\star \omega) \in \text{Pol}_{\frac{n}{2}}(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N \quad (3.18)$$

if n is even. There are no contributions for odd n . Finally, we find that the *odd degree* terms ω_{2j-1}^- contribute through

$$r^{2j-2} {}''\mathcal{H}_1^{p-1} \subset \text{Pol}_{2j-1}^{p-1}(\mathbb{R}^{n-1})$$

and

$$r^{2j-2} i_E({}'\mathcal{H}_1^{\frac{n-1}{2}}) \subset \text{Pol}_{2j-1}^{\frac{n-3}{2}}(\mathbb{R}^{n-1}) \quad \text{and} \quad r^{2j-2} \alpha \wedge {}'\mathcal{H}_0^{\frac{n-3}{2}} \subset \text{Pol}_{2j-1}^{\frac{n-1}{2}}(\mathbb{R}^{n-1})$$

for odd n . Since the module ${}''\mathcal{H}_1^{p-1}$ is isomorphic to $i_E(\mathcal{H}_0^p)$, we find the contributions

$$r^{2j-2} dx_n \wedge i_E(\mathcal{H}_0^p)$$

and

$$r^{2j-2} dx_n \wedge i_E {}'\mathcal{H}_0^{\frac{n+1}{2}} \quad \text{and} \quad r^{2j-2} dx_n \wedge \alpha \wedge {}'\mathcal{H}_0^{\frac{n-3}{2}}$$

for odd n . Altogether, this leads to the homomorphisms

$$\Lambda^p(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \simeq \mathcal{H}_0^p \ni \omega \mapsto x_n^{N-2j+1} r^{2j-2} dx_n \wedge i_E(\omega) \in \text{Pol}_N^p(\mathbb{R}^{n-1}) \quad (3.19)$$

and

$$\Lambda^{\frac{n-1}{2}}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j+1} r^{2j-2} dx_n \wedge i_E(\star \omega) \in \text{Pol}_N^{\frac{n-1}{2}}(\mathbb{R}^{n-1}), \quad (3.20)$$

$$\Lambda^{\frac{n+1}{2}}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j+1} r^{2j-2} dx_n \wedge \alpha \wedge \star \omega \in \text{Pol}_N^{\frac{n+1}{2}}(\mathbb{R}^{n-1}) \quad (3.21)$$

for odd n and $2 \leq 2j \leq N+1$ in the space (3.2).

We summarize the above discussion as

Proposition 3.1.2. *For any $N \in \mathbb{N}_0$, the space (3.2) is generated by*

- *the homomorphisms (3.13), (3.14), (3.19) (for general form degrees p), and*
- *the eight exotic homomorphisms (3.9), (3.10), (3.15), (3.16), (3.17), (3.18), (3.20) and (3.21) (for specific form degrees).*

All these homomorphisms are generated by the operations

- *multiplication by positive integer powers of x_n and r^2 ,*
- *$dx_n \wedge, \alpha \wedge, i_E$ and*
- *\star .*

Any composition of the latter operations yields an element of (3.2) for some $N \in \mathbb{N}_0$. The exotic homomorphisms all contain the Hodge star operator \star as a factor.

Similar arguments can be used to describe the spaces (3.3). Again, we apply the decomposition (3.6) to the Taylor coefficients of $\omega \in \text{Pol}_N^p(\mathbb{R}^n)$ in (3.4). As a result, we find the homomorphisms

$$\Lambda^{p-1}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \simeq \mathcal{H}_0^{p-1} \ni \omega \mapsto x_n^{N-1-2j} r^{2j} \alpha \wedge \omega \in \text{Pol}_N^p(\mathbb{R}^{n-1}) \quad (3.22)$$

for $0 \leq 2j \leq N-1$,

$$\Lambda^{p-1}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \simeq \mathcal{H}_0^{p-1} \ni \omega \mapsto x_n^{N-2j} r^{2j} dx_n \wedge \omega \in \text{Pol}_N^p(\mathbb{R}^{n-1}) \quad (3.23)$$

for $0 \leq 2j \leq N$ and

$$\Lambda^{p-1}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \simeq \mathcal{H}_0^{p-1} \ni \omega \mapsto x_n^{N-2j-2} r^{2j} dx_n \wedge \alpha \wedge i_E(\omega) \in \text{Pol}_N^p(\mathbb{R}^{n-1}) \quad (3.24)$$

for $0 \leq 2j \leq N-2$ and general p . In addition, there are eight exotic homomorphisms for specific form degrees.

A similar discussion yields the following classification of equivariant homomorphisms (3.1) for arbitrary p and q .

Proposition 3.1.3. *For any $N \in \mathbb{N}_0$, the space (3.1) is generated by the homomorphisms of the form (3.13), (3.14) and (3.19) (for $q = p$), the homomorphisms of the form (3.22), (3.23) and (3.24) (for $q = p-1$), the two additional homomorphisms*

$$\Lambda^{p+1}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j-1} r^{2j} i_E(\omega) \in \text{Pol}_N^p(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N-1 \quad (3.25)$$

and

$$\Lambda^{p-2}(\mathbb{R}^{n-1})^* \otimes \mathbb{C} \ni \omega \mapsto x_n^{N-2j-1} r^{2j} dx_n \wedge \alpha \wedge \omega \in \text{Pol}_N^p(\mathbb{R}^{n-1}), \quad 0 \leq 2j \leq N-1 \quad (3.26)$$

as well as the compositions of these homomorphisms with \star . The exotic homomorphisms in (3.2) and (3.3) are restrictions of the latter compositions with \star to appropriate form degrees.

Now we apply these results as follows.

We consider forms on $\mathfrak{n}_-^*(\mathbb{R}) \simeq (\mathbb{R}^n)^*$ with polynomial coefficients. We regard these forms as elements of $\text{Pol}(\mathfrak{n}_-^*(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R}))$. We use coordinates ξ_j on $\mathfrak{n}_-^*(\mathbb{R})$ and the standard bases $\{E_j = E_j^-, j = 1, \dots, n\}$ and $\{E_j^*, j = 1, \dots, n\}$ of $\mathfrak{n}_-(\mathbb{R})$ and $\mathfrak{n}_-^*(\mathbb{R})$.

Since $\mathbb{R}^n \simeq (\mathbb{R}^n)^*$ as $SO(n, \mathbb{R})$ -modules, (3.13), (3.14) and (3.19) imply that there are three types of equivariant homomorphisms in

$$\text{Hom}_{SO(n, \mathbb{R})}(\Lambda^p(\mathfrak{n}_-^*(\mathbb{R})) \otimes \mathbb{C}, \text{Pol}_N(\mathfrak{n}_-^*(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}).$$

These are given by the maps

$$\begin{aligned} \omega &\mapsto \xi_n^{N-2j} |\xi'|^{2j} \otimes \omega, \\ \omega &\mapsto \xi_n^{N-2j-1} |\xi'|^{2j} E_n \wedge i_E(\omega), \\ \omega &\mapsto \xi_n^{N-2j-2} |\xi'|^{2j} \alpha \wedge i_E(\omega) \end{aligned} \quad (3.27)$$

for appropriate values of j . Here we use the definitions

$$\alpha \stackrel{\text{def}}{=} \sum_{j=1}^{n-1} \xi_j \otimes E_j \quad \text{and} \quad i_E \stackrel{\text{def}}{=} \sum_{j=1}^{n-1} \xi_j \otimes i_{E_j^*}.$$

Equivalently, these maps take the form

$$\begin{aligned} E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} &\mapsto \xi_n^{N-2j} |\xi'|^{2j} \otimes E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p}, \\ E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} &\mapsto \xi_n^{N-2j-1} |\xi'|^{2j} \xi_{[\alpha_1} \otimes E_n \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}], \\ E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} &\mapsto \xi_n^{N-2j-2} |\xi'|^{2j} \sum_{j=1}^{n-1} \xi_j \xi_{[\alpha_1} \otimes E_j \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}] \end{aligned} \quad (3.28)$$

on basis elements of \mathcal{H}_0^p given by partitions $1 \leq \alpha_1 < \cdots < \alpha_p \leq n-1$. Here we use the convention

$$\xi_{[\alpha_1} \otimes \omega \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}] = \sum_{l=1}^p (-1)^{l+1} \xi_{\alpha_l} \otimes \omega \otimes E_{\alpha_1} \wedge \cdots \wedge \widehat{E_{\alpha_l}} \wedge \cdots \wedge E_{\alpha_p} \quad (3.29)$$

for any $\omega \in \Lambda^*(\mathfrak{n}_-(\mathbb{R}))$, and we suppress obvious tensor products with copies of \mathbb{C} .

Similarly, by (3.22)–(3.24), there are three types of equivariant homomorphisms

$$\mathrm{Hom}_{SO(\mathfrak{n}'_-(\mathbb{R}))}(\Lambda^{p-1}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}, \mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C})$$

which are given by the maps

$$\begin{aligned} \omega &\mapsto \xi_n^{N-2j} |\xi'|^{2j} \otimes E_n \wedge \omega, \\ \omega &\mapsto \xi_n^{N-2j-1} |\xi'|^{2j} \alpha \wedge \omega, \\ \omega &\mapsto \xi_n^{N-2j-2} |\xi'|^{2j} E_n \wedge \alpha \wedge i_E(\omega) \end{aligned} \quad (3.30)$$

(for appropriate values of j) or, equivalently, by the maps

$$\begin{aligned} E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p-1}} &\mapsto \xi_n^{N-2j} |\xi'|^{2j} \otimes E_n \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p-1}}, \\ E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p-1}} &\mapsto \xi_n^{N-2j-1} |\xi'|^{2j} \sum_{j=1}^{n-1} \xi_j \otimes E_j \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p-1}}, \\ E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p-1}} &\mapsto \xi_n^{N-2j-2} |\xi'|^{2j} \sum_{j=1}^{n-1} \xi_j \xi_{[\alpha_1} \otimes E_n \wedge E_j \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_{p-1}}] \end{aligned} \quad (3.31)$$

on basis elements of \mathcal{H}_0^{p-1} given by partitions $1 \leq \alpha_1 < \cdots < \alpha_{p-1} \leq n-1$. Again, we use the convention (3.29) and we suppress obvious tensor products with copies of \mathbb{C} .

Finally, according to Proposition 3.1.3, there are two additional types

$$\mathrm{Hom}_{SO(\mathfrak{n}'_-(\mathbb{R}))}(\Lambda^{p+1}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}, \mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C})$$

(for $0 \leq p \leq n-2$) and

$$\mathrm{Hom}_{SO(\mathfrak{n}'_-(\mathbb{R}))}(\Lambda^{p-2}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}, \mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C})$$

(for $2 \leq p \leq n$) of homomorphisms which are given by the respective maps

$$\omega \mapsto \xi_n^{N-2j-1} |\xi'|^{2j} i_E(\omega) \quad (3.32)$$

and

$$\omega \mapsto \xi_n^{N-2j-1} |\xi'|^{2j} E_n \wedge \alpha \wedge \omega \quad (3.33)$$

(for appropriate values of j) or, equivalently, by the maps

$$E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p+1}} \mapsto \xi_n^{N-2j-1} |\xi'|^{2j} \xi_{[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_{p+1}}] \quad (3.34)$$

and

$$E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p-2}} \mapsto \xi_n^{N-2j-1} |\xi'|^{2j} \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_{p-2}}. \quad (3.35)$$

Now, by applying the Fourier transform (see Section 2.1), we regard singular vectors which correspond to \mathfrak{g}' -homomorphisms

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^q(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_\eta) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\mu), \quad \eta, \mu \in \mathbb{C}$$

of generalized Verma modules as elements of the spaces

$$\mathrm{Hom}_{\mathfrak{p}'}(\Lambda^q(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_\eta, \mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\mu)$$

for some $N \in \mathbb{N}_0$; here the modules $\Lambda^q(\mathfrak{n}'_-(\mathbb{R}))$ and $\Lambda^p(\mathfrak{n}_-(\mathbb{R}))$ are regarded as trivial A -modules and A acts on $\mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R}))$ by the push-forward action. It follows that

$$\eta = -N + \mu.$$

Therefore, it suffices to study the homomorphisms in the spaces

$$\mathrm{Hom}_{\mathfrak{p}'}(\Lambda^q(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda), \quad \lambda \in \mathbb{C}. \quad (3.36)$$

By Proposition 3.1.3, their \mathfrak{l}' -invariance implies that they are linear combinations of the homogeneous homomorphisms of degree N listed in (3.28), (3.31), (3.34) and (3.35) as well as their compositions with \star . Each choice of $q \in \{p-2, p-1, p, p+1\}$ corresponds to one of these four types of homomorphisms. In order to find explicit formulas for the corresponding singular vectors, it remains to determine those respective linear combinations which, in addition, are annihilated by the operators

$$d\tilde{\pi}_{\lambda,p}(Z), \quad Z \in \mathfrak{n}'_+(\mathbb{R}) \quad (3.37)$$

(see Lemma 2.3.3). This will be the subject of the following subsections. The complete list of singular vectors which correspond to the homomorphisms of the form (3.36) then follows by composing these results with the operator \star . In particular, the discussion will determine *all* singular vectors which describe the embeddings of the $\mathcal{U}(\mathfrak{g}')$ -submodules in Propositions 2.3.1 and 2.3.2.

For the following considerations, it will be convenient to introduce some additional notation. We define the operators

$$P_j(\lambda) \stackrel{\mathrm{def}}{=} \left(\frac{1}{2} \xi_j \Delta_\xi + (\lambda - E_\xi) \partial_{\xi_j} \right) \otimes 1 - \sum_{k=1}^n \partial_{\xi_k} \otimes (E_k^- \otimes E_j^+ - E_j^- \otimes E_k^+) \quad (3.38)$$

for $j = 1, \dots, n-1$ on $\mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes V_{\lambda,p} \simeq \mathrm{Pol}_N(\mathbb{R}^n) \otimes V_{\lambda,p}$. We recall that

$$P_j(\lambda) = id \tilde{\pi}_{\lambda,p}(E_j^+)$$

(see (2.28)). We shall also write ∂_j instead of ∂_{ξ_j} .

3.2. Families of singular vectors of the first type. Assume that $N \in \mathbb{N}_0$. We first consider singular vectors of *odd* homogeneity $2N + 1$ in

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-(2N+1)}, \text{Pol}_{2N+1}(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda),$$

which are linear combinations of the homomorphisms in (3.27). In the following, we shall refer to these singular vectors as singular vectors of the *first* type. They correspond to \mathfrak{g}' -homomorphisms

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-(2N+1)}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda).$$

By (3.27), such singular vectors have the form

$$v_{2N+1}^{(p \rightarrow p)}(\lambda) = \xi_n^{2N+1} P(t) \otimes \text{Id} + \xi_n^{2N} Q(t) E_n \wedge i_E + \xi_n^{2N-1} R(t) \alpha \wedge i_E$$

or, equivalently,

$$\begin{aligned} v_{2N+1}^{(p \rightarrow p)}(\lambda)(E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p}) &= \xi_n^{2N+1} P(t) \otimes E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ &\quad + \xi_n^{2N} Q(t) \xi_{[\alpha_1} \otimes E_n \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \\ &\quad + \xi_n^{2N-1} R(t) \sum_{i=1}^{n-1} \xi_i \xi_{[\alpha_1} \otimes E_i \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \end{aligned} \quad (3.39)$$

for any partition $1 \leq \alpha_1 < \cdots < \alpha_p \leq n-1$. Here

$$t \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{n-1} \xi_i^2}{\xi_n^2} = \frac{|\xi'|^2}{\xi_n^2}$$

and $P(t)$, $Q(t)$, $R(t)$ are polynomials in t of respective degrees N , N and $N-1$ to be determined; the contributions by $R(t)$ do not appear for $N=0$. In order to simplify the notation, we have suppressed obvious tensor products with copies of \mathbb{C} .

We treat ξ_n and t as independent variables. Now using the rules

$$\partial_n = -\frac{2}{\xi_n} t \frac{\partial}{\partial t} \quad \text{and} \quad \partial_j = \frac{2\xi_j}{\xi_n^2} \frac{\partial}{\partial t}, \quad j = 1, \dots, n-1,$$

we find

$$\begin{aligned} \frac{1}{2} \partial_n^2 (\xi_n^{2N+1} P) &= \xi_n^{2N-1} (2t^2 P'' + (1-4N)tP' + N(2N+1)P) \\ \frac{1}{2} \Delta' (\xi_n^{2N+1} P) &= \xi_n^{2N-1} (2tP'' + (n-1)P') \\ \left(\lambda - \sum_{i=1}^n \xi_i \partial_i \right) \partial_j (\xi_n^{2N+1} P) &= \xi_n^{2N-1} \xi_j (2\lambda - 4N) P'. \end{aligned}$$

These results allow to express the action of $P_j(\lambda)$ on

$$v_1 \stackrel{\text{def}}{=} \xi_n^{2N+1} P(t) \otimes E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p}$$

as the sum

$$\begin{aligned} P_j(\lambda) v_1 &= [2t(t+1)P'' + (-4N+1)tP' + (2\lambda+n-4N-1)P' \\ &\quad + N(2N+1)P] \xi_n^{2N-1} \xi_j \otimes E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ &\quad + 2P' \xi_n^{2N-1} \xi_{[\alpha_1} \otimes E_j \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \end{aligned}$$

$$\begin{aligned}
& - 2P' \xi_n^{2N-1} \sum_{i=1}^{n-1} \xi_i \delta_{j[\alpha_1]} \otimes E_i \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}] \\
& + [2tP' - (2N+1)P] \xi_n^{2N} \delta_{j[\alpha_1]} \otimes E_n \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}.
\end{aligned}$$

Similarly, the action of $P_j(\lambda)$ on

$$v_2 \stackrel{\text{def}}{=} \xi_n^{2N} Q(t) \xi_{[\alpha_1]} \otimes E_n \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}$$

is given by

$$\begin{aligned}
P_j(\lambda)v_2 &= [2t(t+1)Q'' + (-4N+3)tQ' + (2\lambda+n-4N+1)Q' \\
& + N(2N-1)Q] \xi_n^{2N-2} \xi_j \xi_{[\alpha_1]} \otimes E_n \wedge E_{\alpha_2} \cdots \wedge E_{\alpha_p} \\
& + (\lambda+p-2N-1)Q \xi_n^{2N} \delta_{j[\alpha_1]} \otimes E_n \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p} \\
& + [2NQ' - 2tQ'] \xi_n^{2N-1} \xi_{[\alpha_1]} \otimes E_j \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}.
\end{aligned}$$

Finally, the action of $P_j(\lambda)$ on

$$v_3 \stackrel{\text{def}}{=} \xi_n^{2N-1} R(t) \sum_{i=1}^{n-1} \xi_i \xi_{[\alpha_1]} \otimes E_i \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}$$

takes the form

$$\begin{aligned}
P_j(\lambda)v_3 &= [2t(t+1)R'' + (-4N+5)tR' + (2\lambda+n-4N+1)R' \\
& + (N-1)(2N-1)R] \xi_n^{2N-3} \xi_j \sum_{i=1}^{n-1} \xi_i \xi_{[\alpha_1]} \otimes E_i \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p} \\
& + [2tR' + (\lambda+n-p-2N)R] \xi_n^{2N-1} \xi_{[\alpha_1]} \otimes E_j \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p} \\
& + (\lambda+p-2N-1)R \xi_n^{2N-1} \sum_{i=1}^{n-1} \xi_i \delta_{j[\alpha_1]} \otimes E_i \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p} \\
& + [2tR' - (2N-1)R] \xi_n^{2N-2} \xi_j \xi_{[\alpha_1]} \otimes E_n \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}.
\end{aligned}$$

We summarize these results and find that the condition

$$P_j(\lambda)v_{2N+1}^{(p \rightarrow p)}(\lambda) = 0 \quad \text{for } j = 1, \dots, n-1$$

is equivalent to the system

$$\begin{aligned}
0 &= 2t(t+1)P'' + (-4N+1)tP' + (2\lambda+n-4N-1)P' + N(2N+1)P, \\
0 &= 2P' + 2NQ - 2tQ' + 2tR' + (\lambda+n-p-2N)R, \\
0 &= -2P' + (\lambda+p-2N-1)R, \\
0 &= 2tP' - (2N+1)P + (\lambda+p-2N-1)Q, \\
0 &= 2t(t+1)Q'' + (-4N+3)tQ' + (2\lambda+n-4N+1)Q' + N(2N-1)Q + 2tR' - (2N-1)R, \\
0 &= 2t(t+1)R'' + (-4N+5)tR' + (2\lambda+n-4N+1)R' + (N-1)(2N-1)R
\end{aligned} \tag{3.40}$$

of ordinary differential equations for P , Q and R .

Now assume that $N \geq 1$ and that the polynomials

$$P(t) = \sum_{j=0}^N p_j t^j, \quad Q(t) = \sum_{j=0}^N q_j t^j \quad \text{and} \quad R(t) = \sum_{j=0}^{N-1} r_j t^j$$

with unknown coefficients solve the system (3.40). We divide the analysis of this assumption into a series of steps. The following arguments will make use of the recursive relations (6.61), (6.62) for (unnormalized) Gegenbauer coefficients.

Step 1. The first equation gives the recurrence relation

$$(N-j+1)(2N-2j+3)p_{j-1} + j(2\lambda+n-4N+2j-3)p_j = 0$$

for $j = 1, \dots, N$. By (6.62), it implies that $p_j = p_j^{(N)}(\lambda)$ are odd Gegenbauer coefficients with $p_N^{(N)}(\lambda)$ being undetermined.

Step 2. The sixth equation gives the recurrence relation

$$((N-1)-j+1)(2(N-1)-2j+3)r_{j-1} + j(2(\lambda-1)+n-4(N-1)+2j-3)r_j = 0$$

for $j = 1, \dots, N-1$. By (6.62), it implies that $r_j = r_j^{(N-1)}(\lambda-1)$ are odd Gegenbauer coefficients with $r_{N-1}^{(N-1)}(\lambda-1)$ being undetermined.

Step 3. Now assume that

$$\lambda+p-2N-1 \neq 0.$$

Then the fourth equation relates $P(t)$ and $Q(t)$ through

$$q_j = \frac{(2N-2j+1)}{(\lambda+p-2N-1)} p_j^{(N)}(\lambda), \quad j = 0, \dots, N. \quad (3.41)$$

Combining this relation with the explicit formula for $p_j^{(N)}(\lambda)$ implies that $q_j = q_j^{(N)}(\lambda-1)$ are even Gegenbauer coefficients with the normalization

$$q_N^{(N)}(\lambda-1) = \frac{1}{(\lambda+p-2N-1)} p_N^{(N)}(\lambda). \quad (3.42)$$

Step 4. Similarly, the third equation relates $P(t)$ and $R(t)$ through

$$r_{j-1} = \frac{2j}{(\lambda+p-2N-1)} p_j^{(N)}(\lambda), \quad j = 1, \dots, N. \quad (3.43)$$

By combining this relation with the explicit formula for $p_j^{(N)}(\lambda)$, we again conclude that $r_j^{(N-1)}(\lambda-1)$ are odd Gegenbauer coefficients with the normalization

$$r_{N-1}^{(N-1)}(\lambda-1) = \frac{2N}{(\lambda+p-2N-1)} p_N^{(N)}(\lambda). \quad (3.44)$$

Step 5. The sum of the second and the third equation gives

$$q_j^{(N)}(\lambda-1) = -\frac{(2\lambda+n-4N+2j-1)}{(2N-2j)} r_j^{(N-1)}(\lambda-1), \quad j = 0, \dots, N-1. \quad (3.45)$$

This relation between $R(t)$ and $Q(t)$ already follows by comparing (3.41) and (3.43) using the recurrence relation (6.62) for $p_j^{(N)}(\lambda)$. In particular, we find the relation

$$2Nq_N^{(N)}(\lambda-1) = r_{N-1}^{(N-1)}(\lambda-1).$$

Step 6. The fifth equation is satisfied by the polynomials determined in the previous steps. In fact, the fifth equation is equivalent to the recurrence relation

$$(N-j)(2N-2j-1)q_j - (2N-2j-1)r_j + (j+1)(2\lambda+n-4N+2j+1)q_{j+1} = 0.$$

By $q_j = q_j^{(N)}(\lambda-1)$, $r_j = r_j^{(N-1)}(\lambda-1)$ and the identity

$$(N-j)(2N-2j-1)q_j^{(N)}(\lambda-1) = -(j+1)(2(\lambda-1)+n-4N+2j+1)q_{j+1}^{(N)}(\lambda-1)$$

(see (6.61)), this relation is equivalent to

$$(2N-2j-1)r_j^{(N-1)}(\lambda-1) = 2(j+1)q_{j+1}^{(N)}(\lambda-1).$$

Again, using (6.61), the latter relation is equivalent to (3.45).

Step 7. If $\lambda+p-2N-1=0$, then the third and the fourth equation are equivalent to $P=0$. Hence the first equation is trivially satisfied. The sixth equations implies $r_j = r_j^{(N-1)}(\lambda-1)$ with an undetermined coefficient $r_{N-1}^{(N-1)}(\lambda-1)$ (as in Step 2). The second equation yields the relation

$$q_j = -\frac{(2\lambda+n-4N+2j-1)}{(2N-2j)}r_j^{(N-1)}(\lambda-1), \quad j = 0, \dots, N-1.$$

It follows that the coefficients q_j are even Gegenbauer coefficients: $q_j = q_j^{(N)}(\lambda-1)$. Then the fifth equation is satisfied by the arguments in Step 6.

For $N=0$, the only solutions are $P=p_0$, $Q=q_0$ with the relation $p_0 = (\lambda+p-1)q_0$.

This completes the analysis of the system (3.40).

Finally, we choose the normalization $p_N^{(N)}(\lambda) = (\lambda+p-2N-1)$ and summarize the above results.

Theorem 3.2.1. *Assume that $N \in \mathbb{N}_0$. The first type homomorphisms*

$$\xi_n^{2N+1}P(t) \otimes \text{Id} + \xi_n^{2N}Q(t)E_n \wedge i_E + \xi_n^{2N-1}R(t)\alpha \wedge i_E$$

with the polynomial coefficients

$$P(t) = \sum_{j=0}^N p_j^{(N)}(\lambda; p)t^j, \quad Q(t) = \sum_{j=0}^N q_j^{(N)}(\lambda-1)t^j \quad \text{and} \quad R(t) = \sum_{j=0}^{N-1} r_j^{(N-1)}(\lambda-1)t^j$$

in the variable $t = |\xi'|^2/\xi_n^2$ are singular vectors of odd homogeneity $2N+1$ in the space

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-(2N+1)}, \text{Pol}_{2N+1}(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

iff

$$\begin{aligned} p_j^{(N)}(\lambda; p) &= (\lambda+p-2N-1)b_j^{(N)}(\lambda), \\ q_j^{(N)}(\lambda) &= a_j^{(N)}(\lambda), \\ r_j^{(N-1)}(\lambda) &= 2Nb_j^{(N-1)}(\lambda), \end{aligned}$$

up to a constant multiple. These singular vectors will be denoted by $v_{2N+1}^{(p \rightarrow p)}(\lambda)$. They describe the embedding of the module

$$\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^p(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-(2N+1)})$$

in Proposition 2.3.1.

We continue with the discussion of singular vectors of *even* homogeneity $2N$, $N \in \mathbb{N}_0$, in

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-2N}, \text{Pol}_{2N}(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda),$$

which are linear combinations of the homomorphisms in (3.27). Again, these will be referred to as singular vectors of the *first type*. They correspond to \mathfrak{g}' -homomorphisms

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-2N}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda).$$

Similarly, as in the case of odd homogeneity, we use the ansatz

$$v_{2N}^{(p \rightarrow p)}(\lambda) = \xi_n^{2N} P(t) \otimes \text{Id} + \xi_n^{2N-1} Q(t) E_n \wedge i_E + \xi_n^{2N-2} R(t) \alpha \wedge i_E$$

or, equivalently,

$$\begin{aligned} v_{2N}^{(p \rightarrow p)}(\lambda)(E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p}) &= \xi_n^{2N} P(t) \otimes E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ &\quad + \xi_n^{2N-1} Q(t) \xi_{[\alpha_1} \otimes E_n \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \\ &\quad + \xi_n^{2N-2} R(t) \sum_{i=1}^{n-1} \xi_i \xi_{[\alpha_1} \otimes E_i \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \end{aligned} \quad (3.46)$$

for any partition $1 \leq \alpha_1 < \cdots < \alpha_p \leq n-1$, where $P(t)$, $Q(t)$ and $R(t)$ are polynomials of respective degrees N , $N-1$ and $N-1$ to be determined.

Analogous calculations as in the case of odd homogeneity show that the condition

$$P_j(\lambda) v_{2N}^{(p \rightarrow p)}(\lambda) = 0 \quad \text{for } j = 1, \dots, n-1$$

is equivalent to the system

$$\begin{aligned} 0 &= 2t(t+1)P'' + (-4N+3)tP' + (2\lambda+n-4N+1)P' + N(2N-1)P, \\ 0 &= 2P' + (2N-1)Q - 2tQ' + 2tR' + (\lambda+n-p-2N+1)R, \\ 0 &= -2P' + (\lambda+p-2N)R, \\ 0 &= 2tP' - 2NP + (\lambda+p-2N)Q, \\ 0 &= 2t(t+1)Q'' + (-4N+5)tQ' + (2\lambda+n-4N+3)Q' + (N-1)(2N-1)Q \\ &\quad + 2tR' - (2N-2)R, \\ 0 &= 2t(t+1)R'' + (-4N+7)tR' + (2\lambda+n-4N+3)R' + (N-1)(2N-3)R \end{aligned} \quad (3.47)$$

of ordinary differential equations for P , Q and R . We omit the details of the calculation. However, we note that although the system (3.47) arises from the system (3.40) by the substitution $N \mapsto N - \frac{1}{2}$ the degrees of the involved polynomials do not coincide.

Now assume that $N \geq 1$ and that the polynomials

$$P(t) = \sum_{j=0}^N p_j t^j, \quad Q(t) = \sum_{j=0}^{N-1} q_j t^j \quad \text{and} \quad R(t) = \sum_{j=0}^{N-1} r_j t^j$$

with unknown coefficients solve the system (3.47). We divide the analysis of this condition into a series of steps. The following arguments are similar to those in the discussion of the system (3.40).

Step 1. By (6.61), the first equation implies that $p_j = p_j^{(N)}(\lambda)$ are even Gegenbauer coefficients with $p_N^{(N)}(\lambda)$ being undetermined.

Step 2. By (6.61), the sixth equation implies that $r_j = r_j^{(N-1)}(\lambda-1)$ are even Gegenbauer coefficients with $r_{N-1}^{(N-1)}(\lambda-1)$ being undetermined.

Step 3. Now assume that

$$\lambda + p - 2N \neq 0.$$

Then the fourth equation relates $P(t)$ and $Q(t)$ through

$$q_j = \frac{(2N-2j)}{(\lambda+p-2N)} p_j^{(N)}(\lambda), \quad j = 0, \dots, N-1. \quad (3.48)$$

By combining this relation with the explicit formula for $p_j^{(N)}(\lambda)$, we conclude that $q_j = q_j^{(N-1)}(\lambda-1)$ are odd Gegenbauer coefficients which are normalized by

$$q_{N-1}^{(N-1)}(\lambda-1) = -\frac{2N(2\lambda-2N+n-1)}{(\lambda+p-2N)} p_N^{(N)}(\lambda).$$

Step 4. The third equation yields the relation

$$r_j^{(N-1)}(\lambda-1) = \frac{2(j+1)}{(\lambda+p-2N)} p_{j+1}^{(N)}(\lambda), \quad j = 0, \dots, N-1. \quad (3.49)$$

In particular, this relates the normalizations of $P(t)$ and $R(t)$ through

$$r_{N-1}^{(N-1)}(\lambda-1) = \frac{2N}{(\lambda+p-2N)} p_N^{(N)}(\lambda).$$

Step 5. The sum of the second and the third equation gives

$$q_j^{(N-1)}(\lambda-1) = -\frac{(2\lambda+n-4N+2j+1)}{(2N-2j-1)} r_j^{(N-1)}(\lambda-1), \quad j = 0, \dots, N-1. \quad (3.50)$$

This relation between $R(t)$ and $Q(t)$ already follows by comparing (3.48) and (3.49) using the recurrence relation (6.61) for $p_j^{(N)}(\lambda)$.

Step 6. The fifth equation is satisfied by the polynomials Q and R determined in the previous steps. In fact, the fifth equation is equivalent to the recurrence relation

$$(N-j-1)(2N-2j-1)q_j + (j+1)(2\lambda+n-4N+2j+3)q_{j+1} + (2j-2N+2)r_j = 0.$$

By $q_j = q_j^{(N-1)}(\lambda-1)$, $r_j = r_j^{(N-1)}(\lambda-1)$ and the identity

$$(N-j-1)(2N-2j-1)q_j^{(N-1)}(\lambda) + (j+1)(2\lambda+n-4N+2j+3)q_{j+1}^{(N-1)}(\lambda) = 0$$

(see (6.62)), this relation is equivalent to

$$(2N-2j-2)r_j^{(N-1)}(\lambda-1) = 2(j+1)q_{j+1}^{(N-1)}(\lambda-1).$$

Again, using (6.62), the latter relation is equivalent to (3.50).

Step 7. If $\lambda + p - 2N = 0$, then the third and the fourth equation are equivalent to $P = 0$. Hence the first equation is trivially satisfied. The sixth equation implies $r_j = r_j^{(N-1)}(\lambda-1)$ with an undetermined coefficient $r_{N-1}^{(N-1)}(\lambda-1)$ (as in Step 2). The second equation yields the relation

$$q_j^{(N-1)}(\lambda-1) = -\frac{(2\lambda+n-4N+2j+1)}{(2N-2j-1)} r_j^{(N-1)}(\lambda-1), \quad j = 0, \dots, N-1.$$

It follows that the coefficients are odd Gegenbauer coefficients: $q_j = q_j^{(N-1)}(\lambda-1)$. The fifth equation is satisfied by the arguments in Step 5.

For $N = 0$, the only solution is $P = p_0$.

This completes the analysis of the system (3.47).

Finally, we choose the normalization $p_N^{(N)}(\lambda) = (\lambda+p-2N)$ and summarize the above results.

Theorem 3.2.2. *Assume that $N \in \mathbb{N}_0$. The first type homomorphisms*

$$\xi_n^{2N} P(t) \otimes \text{Id} + \xi_n^{2N-1} Q(t) E_n \wedge i_E + \xi_n^{2N-2} R(t) \alpha \wedge i_E$$

with the polynomial coefficients

$$P(t) = \sum_{j=0}^N p_j^{(N)}(\lambda; p) t^j, \quad Q(t) = \sum_{j=0}^{N-1} q_j^{(N-1)}(\lambda-1) t^j \quad \text{and} \quad R(t) = \sum_{j=0}^{N-1} r_j^{(N-1)}(\lambda-1) t^j$$

in the variable $t = |\xi'|^2 / \xi_n^2$ are singular vectors of even homogeneity $2N$ in the space

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-2N}, \text{Pol}_{2N}(\mathfrak{n}_-^*(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

iff

$$\begin{aligned} p_j^{(N)}(\lambda; p) &= (\lambda+p-2N) a_j^{(N)}(\lambda), \\ q_j^{(N-1)}(\lambda) &= -2N(2\lambda+n-2N+1) b_j^{(N-1)}(\lambda), \\ r_j^{(N-1)}(\lambda) &= 2N a_j^{(N-1)}(\lambda), \end{aligned}$$

up to a constant multiple. These singular vectors are denoted by $v_{2N}^{(p \rightarrow p)}(\lambda)$. They describe the embedding of the module

$$\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^p(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-2N})$$

in Proposition 2.3.1.

We finish this section with explicit examples of singular vectors of the first type.

Example 3.2.3. *We display the low-homogeneity singular vectors $v_k^{(p \rightarrow p)}(\lambda)$ for $k \leq 3$. The singular vectors of homogeneity 0 is given by*

$$v_0^{(p \rightarrow p)} = (\lambda+p) \otimes \text{Id}.$$

The first type singular vector of homogeneity 1 is given by

$$v_1^{(p \rightarrow p)}(\lambda) = (\lambda+p-1) \xi_n \otimes + E_n \wedge i_E.$$

The first type singular vector of homogeneity 2 is given by

$$\begin{aligned} v_2^{(p \rightarrow p)}(\lambda) &= (\lambda+p-2) \sum_{i=1}^{n-1} \xi_i^2 \otimes \text{Id} - (2\lambda+n-3)(\lambda+p-2) \xi_n^2 \otimes \text{Id} \\ &\quad - 2(2\lambda+n-3) \xi_n E_n \wedge i_E + 2\alpha \wedge i_E. \end{aligned}$$

Finally, the first type singular vector of homogeneity 3 is given by the sum

$$v_3^{(p \rightarrow p)}(\lambda) = (\lambda+p-3) \xi_n \sum_{i=1}^{n-1} \xi_i^2 \otimes \text{Id} - \frac{1}{3}(2\lambda+n-5)(\lambda+p-3) \xi_n^3 \otimes \text{Id}$$

$$+ \sum_{i=1}^{n-1} \xi_i^2 E_n \wedge i_E - (2\lambda + n - 5) \xi_n^2 E_n \wedge i_E + 2\xi_n \alpha \wedge i_E.$$

3.3. Families of singular vectors of the second type. Let $N \in \mathbb{N}$. We consider singular vectors of the form

$$v_N^{(p-1 \rightarrow p)}(\lambda) \in \text{Hom}_{\mathfrak{p}'}(\Lambda^{p-1}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda),$$

which are linear combinations of homomorphisms listed in (3.30). In the following, we shall refer to these as singular vectors of the second type. Such vectors are related to singular vectors $v_N^{(p \rightarrow p)}(\lambda)$ of the first type through conjugation with the Hodge star operators

$$\bar{\star} : \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \rightarrow \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^{n-p}(\mathfrak{n}_-(\mathbb{R}))$$

and

$$\star : \Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \rightarrow \Lambda^{n-1-p}(\mathfrak{n}'_-(\mathbb{R})),$$

i.e.,

$$v_N^{(p-1 \rightarrow p)}(\lambda) = \bar{\star} v_N^{(n-p \rightarrow n-p)}(\lambda) \star.$$

The following two results explicate the structure of singular vectors of the second type. We first describe singular vectors of odd homogeneity.

Theorem 3.3.1. *Assume that $N \in \mathbb{N}_0$. The second type homomorphism*

$$\xi_n^{2N+1} P(t) \otimes E_n + \xi_n^{2N} Q(t) \alpha + \xi_n^{2N-1} R(t) E_n \wedge \alpha \wedge i_E \quad (3.51)$$

with the polynomial coefficients

$$P(t) = \sum_{j=0}^N p_j(\lambda; N, p) t^j, \quad Q(t) = \sum_{j=0}^N q_j^{(N)}(\lambda-1) t^j \quad \text{and} \quad R(t) = \sum_{j=0}^{N-1} r_j^{(N-1)}(\lambda-1) t^j$$

in the variable $t = |\xi'|^2 / \xi_n^2$ are singular vectors of odd homogeneity $2N+1$ in the space

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^{p-1}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-(2N+1)}, \text{Pol}_{2N+1}(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

iff

$$\begin{aligned} p_j(\lambda; N, p) &= -(\lambda + n - p - 2N + 2j - 1) b_j^{(N)}(\lambda), \\ q_j^{(N)}(\lambda) &= a_j^{(N)}(\lambda), \\ r_j^{(N-1)}(\lambda) &= 2N b_j^{(N)}(\lambda). \end{aligned}$$

These singular vectors are denoted by $v_{2N+1}^{(p-1 \rightarrow p)}(\lambda)$. They describes the embedding of the submodule

$$\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^{p-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-(2N+1)})$$

in Proposition 2.3.1.

For singular vectors of even homogeneity we have the following analogous result.

Theorem 3.3.2. *Assume that $N \in \mathbb{N}_0$. The second type homomorphism*

$$\xi_n^{2N} P(t) \otimes E_n + \xi_n^{2N-1} Q(t) \alpha + \xi_n^{2N-2} R(t) E_n \wedge \alpha \wedge i_E \quad (3.52)$$

with the polynomial coefficients

$$P(t) = \sum_{j=0}^N p_j(\lambda; N, p) t^j, \quad Q(t) = \sum_{j=0}^{N-1} q_j^{(N-1)}(\lambda-1) t^j \quad \text{and} \quad R(t) = \sum_{j=0}^{N-1} r_j^{(N-1)}(\lambda-1) t^j$$

in the variable $t = |\xi'|^2 / \xi_n^2$ are singular vectors of even homogeneity $2N$ in the space

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^{p-1}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-2N}, \text{Pol}_{2N}(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

iff

$$\begin{aligned} p_j(\lambda; N, p) &= -(\lambda + n - p - 2N + 2j) a_j^{(N)}(\lambda; p), \\ q_j^{(N-1)}(\lambda) &= -2N(2\lambda + n - 2N + 1) b_j^{(N-1)}(\lambda), \\ r_j^{(N-1)}(\lambda) &= 2N a_j^{(N-1)}(\lambda). \end{aligned}$$

These singular vectors are denoted by $v_{2N}^{(p-1 \rightarrow p)}(\lambda)$. They describe the embedding of the submodule

$$\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^{p-1}(\mathbb{R}^{n-1}) \otimes \mathbb{C}_{\lambda-2N})$$

in Proposition 2.3.1.

There are direct proofs of the latter two results using similar arguments as in Section 3.2 (and resting on the formulas in (3.31)). We omit these details. Both theorems also follow from the results on families of homomorphisms of the first type by conjugation with the Hodge star operators on \mathbb{R}^n and \mathbb{R}^{n-1} . However, we shall not apply this method either. Instead, we shall indirectly verify both results by using the following method. In Section 4, we shall describe the conformal symmetry breaking operators of the first type which are induced by the singular vectors of the first type. By conjugation of these operators with Hodge star operators, we obtain conformal symmetry breaking operators of the second type. These operators correspond to the above singular vectors of the second type, of course.

We finish this section with explicit examples of singular vectors of the second type.

Example 3.3.3. We display the low-homogeneity singular vectors $v_k^{(p-1 \rightarrow p)}(\lambda)$ for $k \leq 3$. The second type singular vector of homogeneity 0 is given by

$$v_0^{(p-1 \rightarrow p)}(\lambda) = -(\lambda + n - p) \otimes E_n.$$

The second type singular vector of homogeneity 1 reads

$$v_1^{(p-1 \rightarrow p)}(\lambda) = -(\lambda + n - p - 1) \xi_n \otimes E_n + \alpha.$$

The second type singular vector of homogeneity 2 is given by

$$\begin{aligned} v_2^{(p-1 \rightarrow p)}(\lambda) &= -(\lambda + n - p) \sum_{i=1}^{n-1} \xi_i^2 \otimes E_n + (2\lambda + n - 3)(\lambda + n - p - 2) \xi_n^2 \otimes E_n \\ &\quad - 2(2\lambda + n - 3) \xi_n \alpha + 2E_n \wedge \alpha \wedge i_E. \end{aligned}$$

Finally, the second type singular vector of homogeneity 3 is given by the sum

$$v_3^{(p-1 \rightarrow p)}(\lambda) = -(\lambda + n - p - 1) \xi_n \sum_{i=1}^{n-1} \xi_i^2 \otimes E_n + \frac{1}{3}(2\lambda + n - 5)(\lambda + n - p - 3) \xi_n^3 \otimes E_n$$

$$+ \sum_{i=1}^{n-1} \xi_i^2 \alpha - (2\lambda + n - 5) \xi_n^2 \alpha + 2\xi_n E_n \wedge \alpha \wedge i_E.$$

3.4. Singular vectors of the third type. Let $N \in \mathbb{N}$ and $1 \leq p \leq n-1$. The singular vectors of the third type in

$$\mathrm{Hom}_{\mathfrak{p}'}(\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \mathrm{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^{p-1}(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

have the form

$$v_N^{(p \rightarrow p-1)} = \xi_n^{N-1} P(t) i_E, \quad (3.53)$$

where

$$P(t) = \sum_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} p_j t^j \quad \text{and} \quad t = \frac{|\xi'|^2}{\xi_n^2}$$

(see (3.32)), i.e.,

$$v_N^{(p \rightarrow p-1)}(E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p}) = \xi_n^{N-1} P(t) \xi_{[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]}.$$

Now assume that the homogeneity is even. In order to compute the actions of the operators $P_j(\lambda)$ (defined in (3.38)) on $v_{2N}^{(p \rightarrow p-1)}$, we calculate

$$\begin{aligned} \frac{1}{2} \xi_j \Delta_\xi (v_{2N}^{(p \rightarrow p-1)}(E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ = [2t(t+1)P'' + (-4N+5)tP' + (n+1)P' + (N-1)(2N-1)P] \\ \times \xi_j \xi_n^{2N-3} \xi_{[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]}, \end{aligned}$$

$$\begin{aligned} (\lambda - E_\xi) \partial_j (v_{2N}^{(p \rightarrow p-1)}(E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ = (\lambda - 2N + 1) [\xi_n^{2N-1} P \delta_{j[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} + 2\xi_j \xi_n^{2N-3} P' \xi_{[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}]] \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^n \partial_k \otimes (E_k^- \otimes E_j^+ - E_j^- \otimes E_k^+) (v_{2N}^{(p \rightarrow p-1)}(E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ = -(p-1) \xi_n^{2N-1} P \delta_{j[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \\ + [(2N-1)P - 2tP'] \xi_n^{2N-2} \xi_{[\alpha_1} \delta_{j[\alpha_2} \otimes E_n \wedge E_{\alpha_3} \wedge \cdots \wedge E_{\alpha_p}]] \\ + 2\xi_n^{2N-3} P' \sum_{k=1}^{n-1} \xi_k \xi_{[\alpha_1} \delta_{j[\alpha_2} \otimes E_k \wedge E_{\alpha_3} \wedge \cdots \wedge E_{\alpha_p}]]. \end{aligned}$$

Hence we conclude

$$\begin{aligned} P_j(\lambda) (v_{2N}^{(p \rightarrow p-1)}(E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ = [2t(t+1)P'' + (-4N+5)tP' + (2\lambda+n-4N+3)P' + (N-1)(2N-1)P] \\ \times \xi_j \xi_n^{2N-3} \xi_{[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \\ + (\lambda + p - 2N) P \xi_n^{2N-1} \delta_{j[\alpha_1} \otimes E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \\ - [(2N-1)P - 2tP'] \xi_n^{2N-2} \xi_{[\alpha_1} \delta_{j[\alpha_2} \otimes E_n \wedge E_{\alpha_3} \wedge \cdots \wedge E_{\alpha_p}]] \end{aligned}$$

$$-2P'\xi_n^{2N-3}\sum_{k=1}^{n-1}\xi_k\xi_{[\alpha_1}\delta_{j[\alpha_2}\otimes E_k\wedge E_{\alpha_3}\wedge\cdots\wedge E_{\alpha_p}]].$$

A similar result holds for $P_j(\lambda)(v_{2N-1}^{(p\rightarrow p-1)})$ (shifting $N \mapsto N - \frac{1}{2}$).

Theorem 3.4.1. *Let $v_N^{(p\rightarrow p-1)}$ be given by (3.53).*

(i) *For $p = 1$, we find the non-trivial singular vector*

$$v_{2N}^{(1\rightarrow 0)} = \xi_n^{2N-1} \left(\sum_{j=0}^{N-1} b_j^{(N-1)} (2N-1)t^j \right) i_E, \quad N \in \mathbb{N}. \quad (3.54)$$

Here $b_j^{(N)}(\lambda)$ are odd Gegenbauer coefficients.

(ii) *For $p = 1$, we find the non-trivial singular vector*

$$v_{2N+1}^{(1\rightarrow 0)} = \xi_n^{2N} \left(\sum_{j=0}^N a_j^{(N)} (2N)t^j \right) i_E, \quad N \in \mathbb{N}_0 \quad (3.55)$$

Here $a_j^{(N)}(\lambda)$ are even Gegenbauer coefficients.

(iii) *For $p = 1, \dots, n-1$, we find the singular vector*

$$v_1^{(p\rightarrow p-1)} = i_E. \quad (3.56)$$

These are all singular vectors of the third type, up to constant multiples.

Proof. We first prove (i). For $p = 1$, the condition $P_j(\lambda)(v_{2N}^{(1\rightarrow 0)}) = 0$ for $j = 1, \dots, n-1$ is equivalent to the system of ordinary differential equation

$$\begin{aligned} 2t(t+1)P'' + (-4N+5)tP' + (2\lambda+n-4N+3)P' + (N-1)(2N-1)P &= 0, \\ (\lambda-2N+1)P &= 0 \end{aligned}$$

for the polynomial $P(t)$. The first equation is satisfied by the polynomial

$$P(t) = \sum_{j=0}^{N-1} b_j^{(N-1)}(\lambda)t^j,$$

where $b_j^{(N)}(\lambda)$ are odd Gegenbauer coefficients. The second equation yields $\lambda = 2N-1$.

We continue with the analogous proof of (ii). For $p = 1$, the condition $P_j(\lambda)(v_{2N+1}^{(1\rightarrow 0)}) = 0$ for $j = 1, \dots, n-1$ is equivalent to the system of ordinary differential equation

$$\begin{aligned} 2t(t+1)P'' + (-4N+3)tP' + (2\lambda+n-4N+1)P' + N(2N-1)P &= 0, \\ (\lambda-2N)P &= 0 \end{aligned}$$

for the polynomial $P(t)$. Again, the first equation is satisfied by the polynomial

$$P(t) = \sum_{j=0}^N a_j^{(N)}(\lambda)t^j,$$

where $a_j^{(N)}(\lambda)$ are even Gegenbauer coefficients. The second equation implies $\lambda = 2N$.

Next, we prove (iii). Let $p = 1, \dots, n-1$. By obvious reasons we find that the condition $P_j(\lambda)(v_1^{(p\rightarrow p-1)}) = 0$ for $j = 1, \dots, n-1$ holds iff $\lambda = -(p-1)$.

Finally, in all other cases there do not exist non-trivial polynomial solutions of the equations $P_j(\lambda)(v_N^{(p \rightarrow p-1)}) = 0$, $j = 1, \dots, n-1$, for any λ . \square

Remark 3.4.2. *The singular vectors in Theorem 3.4.1 can be described in terms of the families of the first type. In fact, it holds*

$$v_{N-1}^{(0 \rightarrow 0)}(N-1)i_E = 0 \quad (3.57)$$

and we have the relation

$$v_N^{(1 \rightarrow 0)} = v_{N-1}^{(0 \rightarrow 0)}(N-1)i_E \quad (3.58)$$

for all $N \in \mathbb{N}$. Here the singular vectors $v_{N-1}^{(0 \rightarrow 0)}(\lambda)$ are defined in Theorem 3.2.1 and Theorem 3.2.2.

Proof. For even homogeneity, the assertions follow from the formulas

$$v_{2N}^{(1 \rightarrow 0)} = \xi_n^{2N-1} \left(\sum_{j=0}^{N-1} b_j^{(N-1)}(2N-1)t^j \right) i_E$$

(see (3.54)) and

$$v_{2N-1}^{(0 \rightarrow 0)}(\lambda) = (\lambda - 2N + 1) \xi_n^{2N-1} \left(\sum_{j=0}^{N-1} b_j^{(N-1)}(\lambda)t^j \right)$$

(by Theorem 3.2.1). Similarly, for odd homogeneity, we use the formulas

$$v_{2N+1}^{(1 \rightarrow 0)} = \xi_n^{2N} \left(\sum_{j=0}^N a_j^{(N)}(2N)t^j \right) i_E$$

(see (3.55)) and

$$v_{2N}^{(0 \rightarrow 0)}(\lambda) = (\lambda - 2N) \xi_n^{2N} \left(\sum_{j=0}^N a_j^{(N)}(\lambda)t^j \right)$$

(by Theorem 3.2.2). \square

Note that (3.57) is a special case of the vanishing formula

$$v_N^{(p \rightarrow p)}(N-p)i_E = 0. \quad (3.59)$$

3.5. Singular vectors of the fourth type. Let $N \in \mathbb{N}$ and $0 \leq p \leq n-2$. The singular vectors of the fourth type in

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^{p+2}(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_\lambda)$$

have the form

$$v_N^{(p \rightarrow p+2)} = \xi_n^{N-1} P(t) E_n \wedge \alpha \quad (3.60)$$

(see (3.33)) with

$$P(t) = \sum_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} p_j t^j \quad \text{and} \quad t = \frac{|\xi'|}{\xi_n^2},$$

i.e.,

$$v_N^{(p \rightarrow p+2)}(E_{\alpha_1} \wedge \dots \wedge E_{\alpha_p}) = \xi_n^{N-1} P(t) \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \dots \wedge E_{\alpha_p}$$

Now assume that the homogeneity is even. In order to compute the action of the operators $P_j(\lambda)$ on $v_{2N}^{(p \rightarrow p+2)}$, we calculate

$$\begin{aligned} & \frac{1}{2} \xi_j \Delta_\xi (v_{2N}^{(p \rightarrow p+2)} (E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ &= [2t(t+1)P'' + (-4N+5)tP' + (n+1)P' + (N-1)(2N-1)P] \\ & \quad \times \xi_j \xi_n^{2N-3} \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p}, \end{aligned}$$

$$\begin{aligned} & (\lambda - E_\xi) \partial_j (v_{2N}^{(p \rightarrow p+2)} (E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ &= (\lambda - 2N + 1) \left[2\xi_j \xi_n^{2N-3} P' \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \right. \\ & \quad \left. + \xi_n^{2N-1} P \otimes E_n \wedge E_j \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \right] \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=1}^n \partial_k \otimes (E_k^- \otimes E_j^+ - E_j^- \otimes E_k^+) (v_{2N}^{(p \rightarrow p+2)} (E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ &= -[2tP' + (n-p-2)P] \xi_n^{2N-1} \otimes E_n \wedge E_j \wedge E_{i_1} \wedge \cdots \wedge E_{i_p} \\ & \quad - [(2N-1)P - 2tP'] \xi_n^{2N-2} \sum_{k=1}^{n-1} \xi_k \otimes E_j \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ & \quad + 2\xi_j \xi_n^{2N-3} P' \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ & \quad - 2\xi_n^{2N-3} P' \sum_{k=1}^{n-1} \xi_k \xi_{[\alpha_1} \otimes E_n \wedge E_k \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p]} \end{aligned}$$

Hence we conclude

$$\begin{aligned} & P_j(\lambda) (v_{2N}^{(p \rightarrow p+2)} (E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p})) \\ &= [2t(t+1)P'' + (-4N+5)tP' + (2\lambda+n-4N+3)P' + (N-1)(2N-1)P] \\ & \quad \times \xi_j \xi_n^{2N-3} \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ & \quad - 2P' \xi_j \xi_n^{2N-3} \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ & \quad + [(\lambda+n-p-2N-1)P + 2tP'] \xi_n^{2N-1} \otimes E_n \wedge E_j \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ & \quad + [(2N-1)P - 2tP'] \xi_n^{2N-2} \sum_{k=1}^{n-1} \xi_k \otimes E_j \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \end{aligned}$$

$$+ 2\xi_n^{2N-3} P' \sum_{k=1}^{n-1} \xi_k \xi_{[\alpha_1]} \otimes E_n \wedge E_k \wedge E_j \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p}.$$

By reasons which will become clear in the proof of the following result, we do not join here the first and the second sum on the right-hand side. A similar computation (shifting $\mathbb{N} \ni N \rightarrow N - \frac{1}{2}$) gives a formula for the action of $P_j(\lambda)$ on $v_{2N-1}^{(p \rightarrow p+2)}$.

Theorem 3.5.1. *Let $v_N^{(p \rightarrow p+2)}$ be given by (3.60).*

(i) *For $p = n - 2$, we find the non-trivial singular vector*

$$v_{2N}^{(n-2 \rightarrow n)} = \xi_n^{2N-1} \left(\sum_{j=0}^{N-1} b_j^{(N-1)} (2N-1) t^j \right) E_n \wedge \alpha, \quad N \in \mathbb{N}. \quad (3.61)$$

Here $b_j^{(N)}(\lambda)$ are odd Gegenbauer coefficients.

(ii) *For $p = n - 2$, we find the non-trivial singular vector*

$$v_{2N+1}^{(n-2 \rightarrow n)} = \xi_n^{2N-1} \left(\sum_{j=0}^N a_j^{(N)} (2N) t^j \right) E_n \wedge \alpha, \quad N \in \mathbb{N}_0. \quad (3.62)$$

Here $a_j^{(N)}(\lambda)$ are even Gegenbauer coefficients.

(iii) *For $p = 0, \dots, n - 2$, we find the singular vector*

$$v_1^{(p \rightarrow p+2)} = E_n \wedge \alpha. \quad (3.63)$$

These are all singular vectors of the fourth type, up to constant multiples.

Proof. We first prove (i). The identity

$$\begin{aligned} & \sum_{k=1}^{n-1} \xi_k \xi_{[\alpha_1]} \otimes E_n \wedge E_k \wedge E_j \wedge E_{\alpha_2} \wedge \cdots \wedge E_{\alpha_p} - \xi_j \sum_{k=1}^{n-1} \xi_k \otimes E_n \wedge E_k \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p} \\ &= - \sum_{k=1}^{n-1} \xi_k^2 \otimes E_n \wedge E_j \wedge E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_p}, \end{aligned}$$

which only holds for the form-degree $p = n - 2$, leads to a cancelation in the above formula for the action of $P_j(\lambda)$ on $v_{2N}^{(n-2 \rightarrow n)}$. It follows that the condition $P_j(\lambda)(v_{2N}^{(n-2 \rightarrow n)}) = 0$ for $j = 1, \dots, n - 1$ is equivalent to the system of ordinary differential equation

$$\begin{aligned} 2t(t+1)P'' + (-4N+5)tP' + (2\lambda+n-4N+3)P' + (N-1)(2N-1)P &= 0, \\ (\lambda-2N+1)P &= 0 \end{aligned}$$

for the polynomial $P(t)$. The first equation is satisfied by the polynomial

$$P(t) = \sum_{j=0}^{N-1} b_j^{(N-1)}(\lambda) t^j,$$

where $b_j^{(N)}(\lambda)$ denote odd Gegenbauer coefficients. The second equation yields $\lambda = 2N - 1$.

We continue with the proof of (ii). By similar arguments as in (1), we find that for $p = n - 2$ the condition $P_j(\lambda)(v_{2N+1}^{(n-2 \rightarrow 2)}) = 0$, $j = 1, \dots, n - 1$, is equivalent to the system of ordinary differential equation

$$\begin{aligned} 2t(t+1)P'' + (-4N+3)tP' + (2\lambda+n-4N+1)P' + N(2N-1)P &= 0, \\ (\lambda-2N)P &= 0 \end{aligned}$$

for the polynomial $P(t)$. The first equation is satisfied by the polynomial

$$P(t) = \sum_{j=0}^N a_j^{(N)}(\lambda)t^j,$$

where $a_j^{(N)}(\lambda)$ are even Gegenbauer coefficients. The second equation fixes $\lambda = 2N$.

Next, we prove (iii). Let $p = 0, \dots, n - 2$. By obvious reasons we find that the condition $P_j(\lambda)(v_1^{(p \rightarrow p+2)}) = 0$ for $j = 1, \dots, n - 1$ holds iff $\lambda = p - n + 2$.

Finally, we note that in all other cases there do not exist non-trivial polynomial solutions of the equations $P_j(\lambda)(v_N^{(p \rightarrow p+2)}) = 0$, $j = 1, \dots, n - 1$, for any λ . \square

Remark 3.5.2. *The singular vectors in Theorem 3.5.1 can be described in terms of the families of the second type. In fact, for all $N \in \mathbb{N}$, it holds*

$$v_{N-1}^{(n-1 \rightarrow n)}(N-1) \wedge \alpha = 0 \quad (3.64)$$

and we have the relation

$$v_N^{(n-2 \rightarrow n)} = -\dot{v}_{N-1}^{(n-1 \rightarrow n)}(N-1) \wedge \alpha \quad (3.65)$$

Here the singular vectors $v_N^{(n-1 \rightarrow n)}(\lambda)$ are defined in Theorem 3.3.1 and Theorem 3.3.2.

Proof. For even homogeneity, we have

$$v_{2N}^{(n-2 \rightarrow n)} = \xi_n^{2N-1} \left(\sum_{j=0}^{N-1} b_j^{(N-1)}(2N-1)t^j \right) E_n \wedge \alpha$$

(by (3.61)) and

$$\begin{aligned} v_{2N-1}^{(n-1 \rightarrow n)}(\lambda) \wedge \alpha &= \sum_{j=0}^{N-1} p_j(\lambda; N-1; n) |\xi'|^{2j} \xi_n^{2N-2j-1} \otimes E_n \wedge \alpha \\ &\quad + \sum_{j=1}^{N-1} r_{j-1}^{(N-2)}(\lambda-1) |\xi'|^{2j} \xi_n^{2N-2j-1} \otimes E_n \wedge \alpha \end{aligned}$$

(by Theorem 3.3.1 and $i_E \wedge \alpha = |\xi'|^2 \otimes \text{Id}$). Now the relations

$$\begin{aligned} p_j(\lambda; N-1; n) &= -(\lambda-2N+2j+1)b_j^{(N-1)}(\lambda), \\ r_{j-1}^{(N-2)}(\lambda-1) &= (2N-2)b_{j-1}^{(N-2)}(\lambda-1) = 2jb_j^{(N-1)}(\lambda) \end{aligned}$$

yield

$$p_j(\lambda; N-1; n) + r_{j-1}^{(N-2)}(\lambda-1) = -(\lambda-2N+1)b_j^{(N-1)}(\lambda).$$

Hence

$$\begin{aligned} v_{2N-1}^{(n-1 \rightarrow n)}(\lambda) \wedge \alpha &= -(\lambda - 2N + 1) \sum_{j=0}^{N-1} b_j^{(N-1)}(\lambda) |\xi'|^{2j} \xi_n^{2N-2j-1} \otimes E_n \wedge \alpha \\ &= -(\lambda - 2N + 1) \xi_n^{2N-1} \left(\sum_{j=0}^{N-1} b_j^{(N-1)}(\lambda) t^j \right) E_n \wedge \alpha. \end{aligned}$$

This proves the claims. Analogous arguments apply for odd homogeneity using Theorem 3.3.2. We omit the details. \square

Note that (3.64) is a special case of the vanishing result

$$v_N^{(p-1 \rightarrow p)}(N - n + p) \alpha = 0 \quad (3.66)$$

Finally, we note that the results of this section also follow from those in Section 3.4 by a Hodge star conjugation argument. This is analogous to the relation between the results in Section 3.2 and Section 3.3.

3.6. Middle degree cases. Here we describe the singular vectors which describe the embeddings of the submodules in Proposition 2.3.2.

Case 1a: Let n be odd and $p = \frac{n-1}{2}$.

The $SO(\mathfrak{n}'_-(\mathbb{R}))$ -module $\Lambda^{\frac{n-1}{2}}(\mathfrak{n}'_-(\mathbb{R}))$ is not irreducible and the explicit formulas for the singular vectors

$$v_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2})}(\lambda)$$

of the first type, which are defined in Theorem 3.2.1 (for odd N) and Theorem 3.2.2 (for even N), show that their restrictions

$$v_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda)$$

to the subspaces

$$\Lambda_{\pm}^{\frac{n-1}{2}}(\mathfrak{n}'_-(\mathbb{R})) \subset \Lambda^{\frac{n-1}{2}}(\mathfrak{n}'_-(\mathbb{R}))$$

are non-trivial. Then the space

$$\text{Hom}_{\mathfrak{p}'}(\Lambda_{\pm}^{\frac{n-1}{2}}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^{\frac{n-1}{2}}(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda})$$

is generated by $v_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda)$. These singular vectors describe the embeddings of the modules in the first two sums in the decomposition (2.24).

The singular vectors which describe the embeddings of the modules in the last sum in the decomposition (2.24) are given by homomorphisms of the second type.

Case 1b: Let n be odd and $p = \frac{n+1}{2}$.

The space

$$\text{Hom}_{\mathfrak{p}'}(\Lambda_{\pm}^{\frac{n+1}{2}}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^{\frac{n+1}{2}}(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda})$$

is generated by the homomorphism

$$\bar{\star} v_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda) \sim v_N^{(\frac{n-1}{2} \rightarrow \frac{n+1}{2}), \pm}(\lambda), \quad (3.67)$$

where $\bar{\star}$ denotes the Hodge star operator on $\mathfrak{n}_-(\mathbb{R}) \simeq \mathbb{R}^n$ with the Euclidean metric. These singular vectors describe the embeddings of the modules in the last two sums in the decomposition (2.25).

The singular vectors which describe the embeddings of the modules in the first sum of the decomposition (2.25) are given by Theorem 3.2.1 (for odd N) and Theorem 3.2.2 (for even N).

Case 2: Let n be even and $p = \frac{n}{2}$.

The $SO(\mathfrak{n}_-(\mathbb{R}))$ -module $\Lambda^{\frac{n}{2}}(\mathfrak{n}_-(\mathbb{R}))$ is not irreducible and the spaces

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^{\frac{n}{2}}(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^{\frac{n}{2}}_+(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda})$$

are generated by the projections

$$v_N^{(\frac{n}{2} \rightarrow \frac{n}{2}), \pm}(\lambda) \stackrel{\text{def}}{=} pr_{\pm}(v_N^{(\frac{n}{2} \rightarrow \frac{n}{2})}(\lambda)) \quad (3.68)$$

of the singular vectors

$$v_N^{(\frac{n}{2} \rightarrow \frac{n}{2})}(\lambda)$$

which are defined in Theorem 3.2.1 (for odd N) and Theorem 3.2.2 (for even N) to the subspaces

$$\Lambda^{\frac{n}{2}}_{\pm}(\mathfrak{n}_-(\mathbb{R})) \subset \Lambda^{\frac{n}{2}}(\mathfrak{n}_-(\mathbb{R}))$$

Here $pr_{\pm} : \Lambda^{\frac{n}{2}}(\mathfrak{n}_-(\mathbb{R})) \rightarrow \Lambda^{\frac{n}{2}}_{\pm}(\mathfrak{n}_-(\mathbb{R}))$ denotes the projections onto the eigenspaces of the Hodge star operator. These singular vectors describe the embeddings of the modules in the decomposition (2.26).

4. CONFORMAL SYMMETRY BREAKING OPERATORS ON DIFFERENTIAL FORMS

In this section, we translate the four types of homomorphisms of generalized Verma modules constructed in Section 3 into four types of conformal symmetry breaking operators

$$D_N^{(p \rightarrow q)} : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^q(\mathbb{R}^{n-1})$$

of order $N \in \mathbb{N}$ on differential forms. These operators will be referred to as operators of the first, second, third and fourth type, respectively.

We first fix some conventions. As before, we consider \mathbb{R}^{n-1} as a codimension subspace of \mathbb{R}^n by

$$\iota : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n, \quad (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 0).$$

Then d , δ and \bar{d} , $\bar{\delta}$ denote the respective differentials and co-differentials on differential forms on \mathbb{R}^{n-1} and \mathbb{R}^n , respectively. Let $\{e_i\}_{i=1}^{n-1}$ denote the standard orthonormal basis on the Euclidean space \mathbb{R}^{n-1} and let ∂_i denote the partial derivative in the i^{th} coordinate. We have

$$d\omega(X_0, \dots, X_p) = \sum_{i=0}^p (-1)^i X_i \left(\omega(X_0, \dots, \widehat{X}_i, \dots, X_p) \right)$$

and

$$\delta\omega(X_1, \dots, X_{p-1}) = - \sum_{i=1}^{n-1} \partial_i \omega(e_i, X_1, \dots, X_{p-1})$$

for $\omega \in \Omega^p(\mathbb{R}^{n-1})$ and smooth vector fields $X_j \in \mathfrak{X}(\mathbb{R}^{n-1})$.

The Laplacian on forms on \mathbb{R}^{n-1} is defined by $\Delta = \delta d + d\delta$. Similarly, we set

$$\bar{\Delta} = \bar{\delta} \bar{d} + \bar{d} \bar{\delta} : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^n).$$

Note that $\Delta = - \sum_{k=1}^{n-1} \partial_k^2$ and $\bar{\Delta} = - \sum_{k=1}^n \partial_k^2$.

We shall consider d and δ also as operators on forms on \mathbb{R}^n . In particular, Δ will also be viewed as an operator acting on $\Omega^*(\mathbb{R}^n)$.

The insertion operator, given by contracting the first form index by the vector field $X \in \mathfrak{X}(\mathbb{R}^n)$, is denoted by $i_X : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^n)$.

The following lemma collects identities which will be used later on.

Lemma 4.0.1. *The following relations hold true as identities on forms on \mathbb{R}^n .*

- (1) $\iota^* \bar{d} = d\iota^*$ and $i_{\partial_n} \bar{\delta} = -\delta i_{\partial_n}$,
- (2) $i_{\partial_n} \partial_n = \delta - \bar{\delta}$ and $\iota^* \partial_n = \iota^* i_{\partial_n} \bar{d} + d\iota^* i_{\partial_n}$,
- (3) $\partial_n^2 = \Delta - \bar{\Delta}$.

Moreover, ∂_n commutes with d , δ and i_{∂_n} .

We emphasize that the second relation in Lemma 4.0.1/(1), the first relation in (2) and the relation (3) hold true as identities on \mathbb{R}^n , i.e., off the subspace \mathbb{R}^{n-1} .

Proof. (1) is trivial. The first identity in (2) is obvious. Cartan's formula $\mathcal{L}_X = i_X d + d i_X$ applied to the normal vector field $X = \partial_n$ yields $\partial_n = i_{\partial_n} \bar{d} + \bar{d} i_{\partial_n}$. Hence $\iota^* \partial_n = \iota^* i_{\partial_n} \bar{d} + \iota^* \bar{d} i_{\partial_n}$. This implies the second relation in (2) using the first relation in (1). (3) is obvious by the respective formulas for the Laplace operators. \square

Now, we use the dual pairing of generalized Verma modules and induced representations [KP14], [KOSS15] to translate the homomorphisms constructed in Section 3 into differential operators:

$$\Omega^p(\mathfrak{n}_-(\mathbb{R})) \rightarrow \Omega^q(\mathfrak{n}'_-(\mathbb{R})).$$

By combining these operators with the identifications $\mathfrak{n}_-(\mathbb{R}) \simeq \mathbb{R}^n$ and $\mathfrak{n}'_-(\mathbb{R}) \simeq \mathbb{R}^{n-1}$ given by the basis vectors E_j , we obtain conformal symmetry breaking operators $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^q(\mathbb{R}^{n-1})$.

The following lemma collects important information on the translation of basic operations on polynomial differential forms into operators on differential forms. We recall the notation $\alpha = \sum_{j=1}^{n-1} \xi_j \otimes E_j$ and $i_E = \sum_{j=1}^{n-1} \xi_j \otimes i_{E_j^*}$ (Section 3.1).

Lemma 4.0.2. *The dualization sends*

$$E_n \wedge \mapsto i_{\partial_n}, \quad \xi_j \mapsto i \partial_j \quad \text{and} \quad i_E \mapsto id, \quad \alpha \wedge \mapsto -i \delta.$$

Proof. The first two claims are obvious. In order to prove the third rule, it suffices to verify that the adjoint map of the principal symbol of id corresponds to the insertion i_E . But the principal symbol of id (regarded as an operator on $\Omega^p(\mathfrak{n}'_-(\mathbb{R}))$) is given by

$$\sum_{j=1}^{n-1} \xi_j \otimes E_j^* \wedge : \Lambda^p(\mathfrak{n}'_-(\mathbb{R}))^* \rightarrow \Lambda^{p+1}(\mathfrak{n}'_-(\mathbb{R}))^*.$$

Its adjoint equals

$$i_E = \sum_{j=1}^{n-1} \xi_j \otimes i_{E_j^*} : \Lambda^{p+1}(\mathfrak{n}'_-(\mathbb{R})) \rightarrow \Lambda^p(\mathfrak{n}'_-(\mathbb{R})).$$

Similarly, the principal symbol of $-i\delta$ (regarded as an operator on $\Omega^p(\mathfrak{n}'_-(\mathbb{R}))$) is given by $\sum_{j=1}^{n-1} \xi_j \otimes i_{E_j} : \Lambda^p(\mathfrak{n}'_-(\mathbb{R}))^* \rightarrow \Lambda^{p-1}(\mathfrak{n}'_-(\mathbb{R}))^*$. Its adjoint equals

$$\sum_{j=1}^{n-1} \xi_j \otimes E_j \wedge = \alpha \wedge : \Lambda^{p-1}(\mathfrak{n}'_-(\mathbb{R})) \rightarrow \Lambda^p(\mathfrak{n}'_-(\mathbb{R})).$$

The proof is complete. \square

Example 4.0.3. *As an illustration, we show how the Fourier transform maps first type second-order homogeneous homomorphisms (encapsulated by singular vectors) into second-order differential operators $\Omega^*(\mathbb{R}^n) \rightarrow \Omega^*(\mathbb{R}^{n-1})$. By Example 3.2.3, the $\mathcal{U}(\mathfrak{g}')$ -homomorphism of generalized Verma modules*

$$\mathcal{M}_{\mathfrak{p}'}^{\mathfrak{g}'}(\Lambda^p(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-2}) \rightarrow \mathcal{M}_{\mathfrak{p}}^{\mathfrak{g}}(\Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda})$$

is induced by

$$\begin{aligned} \omega &\mapsto (\lambda+p-2)|\xi'|^2 \otimes \omega \\ &\quad - (2\lambda+n-3)(\lambda+p-2)\xi_n^2 \otimes \omega \\ &\quad - 2(2\lambda+n-3)\xi_n E_n \wedge i_E(\omega) + 2\alpha \wedge i_E(\omega). \end{aligned}$$

Now, using Lemma 4.0.2, this homomorphism translates into the differential operator

$$\begin{aligned} \Omega^p(\mathbb{R}^n) \ni \omega &\mapsto D_2^{(p \rightarrow p)}(\lambda)(\omega) = (\lambda+p-2)\Delta \iota^* \omega \\ &\quad + (2\lambda+n-3)(\lambda+p-2)\iota^* \partial_n^2(\omega) \\ &\quad + 2(2\lambda+n-3)d\iota^* i_{\partial_n} \partial_n \omega + 2d\delta \iota^* \omega \in \Omega^p(\mathbb{R}^{n-1}). \end{aligned}$$

4.1. Families of the first type. We recall that $a_j^{(N)}(\lambda)$ and $b_j^{(N)}(\lambda)$ denote even and odd Gegenbauer coefficients, respectively (see Appendix).

Theorem 4.1.1. *Assume that $N \in \mathbb{N}$ and $p = 0, \dots, n-1$. Then the family*

$$D_{2N}^{(p \rightarrow p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C}$$

of differential operators of order $2N$ which is defined by the formula

$$\begin{aligned} D_{2N}^{(p \rightarrow p)}(\lambda) &= \sum_{j=0}^N (-1)^{N-j} p_j^{(N)}(\lambda; p) \Delta^j \iota^* \partial_n^{2N-2j} \\ &\quad + \sum_{j=0}^{N-1} (-1)^{N-j} q_j^{(N-1)}(\lambda-1) \Delta^j d\iota^* i_{\partial_n} \partial_n^{2N-2j-1} \\ &\quad + \sum_{j=0}^{N-1} (-1)^{N-j-1} r_j^{(N-1)}(\lambda-1) \Delta^j d\delta \iota^* \partial_n^{2N-2j-2}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} p_j^{(N)}(\lambda; p) &= (\lambda+p-2N)a_j^{(N)}(\lambda), \\ q_j^{(N-1)}(\lambda) &= -2N(2\lambda+n-2N+1)b_j^{(N-1)}(\lambda), \\ r_j^{(N-1)}(\lambda) &= 2Na_j^{(N-1)}(\lambda) \end{aligned}$$

is infinitesimally equivariant in the sense that

$$D_{2N}^{(p \rightarrow p)}(\lambda) d\pi_{\lambda,p}^{\vee}(X) = d\pi'_{\lambda-2N,p}(X) D_{2N}^{(p \rightarrow p)}(\lambda), \quad X \in \mathfrak{g}'(\mathbb{R}). \quad (4.2)$$

Proof. The singular vector $v_{2N}^{(p \rightarrow p)}(\lambda)$ in Theorem 3.2.2 corresponds to an operator with the equivariance as in (4.2). The explicit formula for the operator follows by combining the explicit formula for the singular vector with Lemma 4.0.2. \square

The intertwining relations (4.8) and (4.2) can also be stated in terms of the geometrically defined representations

$$\pi_{\lambda}^{(p)}(\gamma) \stackrel{\text{def}}{=} e^{\lambda \Phi_{\gamma}} \gamma_* : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^n) \quad (4.3)$$

and their analogs $\pi_{\lambda}^{\prime(p)}$ on \mathbb{R}^{n-1} . Here γ denotes conformal diffeomorphisms of the Euclidean metric g_0 on \mathbb{R}^n , i.e., $\gamma_*(g_0) = e^{2\Phi_{\gamma}} g_0$ for some $\Phi_{\gamma} \in C^{\infty}(\mathbb{R}^n)$. In order to obtain such a formulation, one has to use the relation

$$\pi_{-\lambda-p}^{(p)} = \pi_{\lambda,p}^{\vee}. \quad (4.4)$$

The identity (4.4) is the non-compact analog of (2.18). In particular, we find that

$$D_{2N}^{(p \rightarrow p)}(\lambda) d\pi_{-\lambda-p}^{(p)}(X) = d\pi'_{-\lambda+2N-p}(X) D_{2N}^{(p \rightarrow p)}(\lambda), \quad X \in \mathfrak{g}'(\mathbb{R}).$$

This corresponds to the formulation in Section 1 (see (1.5)).

We continue with the formulation of the analogous result for *odd* order conformal symmetry breaking operators of the first type.

Theorem 4.1.2. *Let $N \in \mathbb{N}_0$ and $p = 0, \dots, n-1$. The family*

$$D_{2N+1}^{(p \rightarrow p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C}$$

of differential operators of order $2N+1$ which is defined by the formula

$$\begin{aligned} D_{2N+1}^{(p \rightarrow p)}(\lambda) &= \sum_{j=0}^N (-1)^{N-j} p_j^{(N)}(\lambda; p) \Delta^j \iota^* \partial_n^{2N+1-2j} \\ &\quad + \sum_{j=0}^N (-1)^{N-j} q_j^{(N)}(\lambda-1) \Delta^j d\iota^* i_{\partial_n} \partial_n^{2N-2j} \\ &\quad + \sum_{j=0}^{N-1} (-1)^{N-j-1} r_j^{(N-1)}(\lambda-1) \Delta^j d\delta \iota^* \partial_n^{2N-1-2j}, \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} p_j^{(N)}(\lambda; p) &= (\lambda + p - 2N - 1) b_j^{(N)}(\lambda), \\ q_j^{(N)}(\lambda) &= a_j^{(N)}(\lambda), \\ r_j^{(N-1)}(\lambda) &= 2N b_j^{(N-1)}(\lambda) \end{aligned}$$

is infinitesimally equivariant in the sense that

$$D_{2N+1}^{(p \rightarrow p)}(\lambda) d\pi_{\lambda,p}^{\vee}(X) = d\pi'_{\lambda-2N-1,p}(X) D_{2N+1}^{(p \rightarrow p)}(\lambda), \quad X \in \mathfrak{g}'(\mathbb{R}). \quad (4.6)$$

Proof. The proof is parallel to that of Theorem 4.1.2. The operator $D_{2N+1}^{(p \rightarrow p)}(\lambda)$ is induced by the singular vector in Theorem 3.2.1. ⁹ \square

Similarly as for even-order families, the intertwining relations (4.10) and (4.6) can also be stated in terms of the geometrically defined representations (4.3) and their analogs on \mathbb{R}^{n-1} . In particular, we find that

$$D_{2N+1}^{(p \rightarrow p)}(\lambda) d\pi_{-\lambda-p}^{(p)}(X) = d\pi'_{-\lambda+2N+1-p}^{(p)}(X) D_{2N+1}^{(p \rightarrow p)}(\lambda), \quad X \in \mathfrak{g}'(\mathbb{R}).$$

This corresponds to the formulation in Section 1 (see (1.11)).

4.2. Families of the second type. The families $D_N^{(p \rightarrow p)}(\lambda)$ of the first type have natural counterparts which map $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^{n-1})$. These are the conformal symmetry breaking operators of the second type. We start with the description of the even-order case.

Theorem 4.2.1. *Assume that $N \in \mathbb{N}$ and $p = 1, \dots, n$. Then the family*

$$D_{2N}^{(p \rightarrow p-1)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C}$$

of differential operators of order $2N$ which is defined by the formula

$$\begin{aligned} D_{2N}^{(p \rightarrow p-1)}(\lambda) = & \sum_{j=0}^N (-1)^{N-j} p_j(\lambda; N, p) \Delta^j \iota^* i_{\partial_n} \partial_n^{2N-2j} \\ & + \sum_{j=0}^{N-1} (-1)^{N-j-1} q_j^{(N-1)}(\lambda-1) \Delta^j \delta \iota^* \partial_n^{2N-1-2j} \\ & + \sum_{j=0}^{N-1} (-1)^{N-j-1} r_j^{(N-1)}(\lambda-1) \Delta^j d \delta \iota^* i_{\partial_n} \partial_n^{2N-2-2j}, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} p_j(\lambda; N, p) &= -(\lambda + n - p - 2N + 2j) a_j^{(N)}(\lambda), \\ q_j^{(N-1)}(\lambda) &= -2N(2\lambda + n - 2N + 1) b_j^{(N-1)}(\lambda), \\ r_j^{(N-1)}(\lambda) &= 2N a_j^{(N-1)}(\lambda) \end{aligned}$$

is infinitesimally equivariant in the sense that

$$D_{2N}^{(p \rightarrow p-1)}(\lambda) d\pi_{\lambda, p}^{\vee}(X) = d\pi'_{\lambda-2N, p-1}^{\vee}(X) D_{2N}^{(p \rightarrow p-1)}(\lambda), \quad X \in \mathfrak{g}'(\mathbb{R}). \quad (4.8)$$

The odd-order analog of Theorem 4.2.1 reads as follows.

Theorem 4.2.2. *Let $N \in \mathbb{N}_0$ and $p = 1, \dots, n$. The family*

$$D_{2N+1}^{(p \rightarrow p-1)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C}$$

of differential operators of order $2N+1$ which is defined by the formula

$$D_{2N+1}^{(p \rightarrow p-1)}(\lambda) = \sum_{j=0}^N (-1)^{N-j} p_j(\lambda; N, p) \Delta^j \iota^* i_{\partial_n} \partial_n^{2N+1-2j}$$

⁹In the final formulas we omit a factor of $-i$.

$$\begin{aligned}
& + \sum_{j=0}^N (-1)^{N-j-1} q_j^{(N)} (\lambda-1) \Delta^j \delta \iota^* \partial_n^{2N-2j} \\
& + \sum_{j=0}^{N-1} (-1)^{N-j-1} r_j^{(N-1)} (\lambda-1) \Delta^j d \delta \iota^* i_{\partial_n} \partial_n^{2N-1-2j},
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
p_j(\lambda; N, p) &= -(\lambda + n - p - 2N + 2j - 1) b_j^{(N)}(\lambda), \\
q_j^{(N)}(\lambda) &= a_j^{(N)}(\lambda), \\
r_j^{(N-1)}(\lambda) &= 2N b_j^{(N-1)}(\lambda)
\end{aligned}$$

is infinitesimally equivariant in the sense that

$$D_{2N+1}^{(p \rightarrow p-1)}(\lambda) d\pi_{\lambda, p}^{\vee}(X) = d\pi_{\lambda-2N-1, p-1}^{\vee}(X) D_{2N+1}^{(p \rightarrow p-1)}(\lambda), \quad X \in \mathfrak{g}'(\mathbb{R}). \tag{4.10}$$

The proofs of Theorem 4.2.1 and Theorem 4.2.2 for families of the second type follow from the corresponding results in Section 4.1 for families of the first type by conjugation with Hodge star operators. The details of this argument will be given in Section 4.3.

4.3. Hodge conjugation. We first recall some well-known general facts.

Lemma 4.3.1. *On any Riemannian manifold (M^n, g) , the differential operators d , δ , Δ and the Hodge star operator \star satisfy the relations*

- (1) $\star\star = (-1)^{p(n-p)}$,
- (2) $\star d\star = (-1)^{n(p+1)+1} \delta$ and $\star d = \delta \star (-1)^{p+1}$,
- (3) $\star \delta \star = (-1)^{n(p+1)} d$ and $\star \delta = d \star (-1)^p$,
- (4) $\star d\delta = \delta d\star$, $\star \delta d = d\delta\star$ and $\star \Delta = \Delta \star$

on $\Omega^p(M)$.

Proof. We omit the proof of (1) and of the first relation in (2). (1) and the first claim in (2) imply the second claims in (2) and (3). Again, this yields the first claim in (3) by using (1). The identities in (4) are immediate consequences. \square

Now let \star and $\bar{\star}$ be the respective Hodge star operators on $\Omega^*(\mathbb{R}^{n-1})$ and $\Omega^*(\mathbb{R}^n)$.

Lemma 4.3.2. *We have the identities*

$$\begin{aligned}
\star \circ \iota^* i_{\partial_n} \circ \bar{\star} &= \iota^* (-1)^{(p+1)(n-1)}, \\
\star \circ \iota^* \circ \bar{\star} &= \iota^* i_{\partial_n} (-1)^{pn+1}
\end{aligned}$$

on $\Omega^p(\mathbb{R}^n)$.

Proof. The second claim is easy to see. We omit the details. The first claim follows by applying Lemma 4.3.1/(1) to the second claim. \square

The following theorem is the main result of the present section.

Theorem 4.3.3. *Let $N \in \mathbb{N}$ and $p = 1, \dots, n$. Then the even-order families $D_{2N}^{(p \rightarrow p)}(\lambda)$ of the first type are Hodge conjugate to the even-order families $D_{2N}^{(p \rightarrow p-1)}(\lambda)$ of the second type. More precisely, we have*

$$D_{2N}^{(p \rightarrow p-1)}(\lambda) = (-1)^{pn} \star D_{2N}^{(n-p \rightarrow n-p)}(\lambda) \bar{\star}. \tag{4.11}$$

Similarly, for $N \in \mathbb{N}_0$, the odd-order families $D_{2N+1}^{(p \rightarrow p)}(\lambda)$ of the first type are Hodge conjugate to the odd-order families $D_{2N+1}^{(p \rightarrow p-1)}(\lambda)$ of the second type. More precisely, we have

$$D_{2N+1}^{(p \rightarrow p-1)}(\lambda) = (-1)^{pn} \star D_{2N+1}^{(n-p \rightarrow n-p)}(\lambda) \bar{\star}. \quad (4.12)$$

Proof. On the one hand, Theorem 4.1.1 implies

$$\begin{aligned} D_{2N}^{(p \rightarrow p)}(\lambda) &= \sum_{j=1}^N (-1)^{N-j} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^j \iota^* \partial_n^{2N-2j} \\ &\quad + \sum_{j=0}^N (-1)^{N-j} p_j^{(N)}(\lambda; p) (\delta d)^j \iota^* \partial_n^{2N-2j} \\ &\quad + \sum_{j=0}^{N-1} (-1)^{N-j} q_j^{(N-1)}(\lambda-1) (d\delta)^j d\iota^* i_{\partial_n} \partial_n^{2N-1-2j} \end{aligned}$$

with the coefficients

- $p_j^{(N)}(\lambda; p) = (\lambda + p - 2N) a_j^{(N)}(\lambda)$,
- $q_j^{(N-1)}(\lambda-1) = -2N(2\lambda + n - 2N - 1) b_j^{(N-1)}(\lambda-1)$,
- $r_j^{(N-1)}(\lambda-1) = 2N a_j^{(N-1)}(\lambda-1)$.

On the other hand, Theorem 4.2.1 shows that

$$\begin{aligned} D_{2N}^{(p \rightarrow p-1)}(\lambda) &= \sum_{j=0}^N (-1)^{N-j} \left[p_j(\lambda; N, p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^j \iota^* i_{\partial_n} \partial_n^{2N-2j} \\ &\quad + \sum_{j=1}^N (-1)^{N-j} p_j(\lambda; N, p) (\delta d)^j \iota^* i_{\partial_n} \partial_n^{2N-2j} \\ &\quad + \sum_{j=0}^{N-1} (-1)^{N-j-1} q_j^{(N-1)}(\lambda-1) (\delta d)^j \delta \iota^* \partial_n^{2N-1-2j}, \end{aligned}$$

with the coefficients

- $p_j(\lambda; N, p) = -(\lambda + n - p - 2N + 2j) a_j^{(N)}(\lambda)$,
- $q_j^{(N-1)}(\lambda-1) = -2N(2\lambda + n - 2N - 1) b_j^{(N-1)}(\lambda-1)$,
- $r_j^{(N-1)}(\lambda-1) = 2N a_j^{(N-1)}(\lambda-1)$.

Now, since ∂_n commutes with $\bar{\star}$, Lemma 4.3.1 and Lemma 4.3.2 imply

$$\begin{aligned} \star (d\delta)^j \iota^* \bar{\star} &= (\delta d)^j \star \iota^* \bar{\star} = (\delta d)^j \iota^* i_{\partial_n} (-1)^{pn+1}, \\ \star (\delta d)^j \iota^* \bar{\star} &= (\delta d)^j \star \iota^* \bar{\star} = (\delta d)^j \iota^* i_{\partial_n} (-1)^{pn+1} \end{aligned}$$

and

$$\star (d\delta)^j d\iota^* i_{\partial_n} \bar{\star} = (\delta d)^j \star d\iota^* i_{\partial_n} \bar{\star} = (\delta d)^j \delta \star \iota^* i_{\partial_n} \bar{\star} (-1)^{n-p} = (\delta d)^j \delta \iota^* (-1)^{pn+1}$$

for $j \in \mathbb{N}_0$. Hence we find

$$(-1)^{pn+1} \star D_{2N}^{(n-p \rightarrow n-p)}(\lambda) \bar{\star}$$

$$\begin{aligned}
&= \sum_{j=1}^N (-1)^{N-j} \left[p_j^{(N)}(\lambda; n-p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (\delta d)^j \iota^* i_{\partial_n} \partial_n^{2N-2j} \\
&+ \sum_{j=0}^N (-1)^{N-j} p_j^{(N)}(\lambda; n-p) (\delta d)^j \iota^* i_{\partial_n} \partial_n^{2N-2j} \\
&+ \sum_{j=0}^{N-1} (-1)^{N-j} q_j^{(N-1)}(\lambda-1) (\delta d)^j \delta \iota^* \partial_n^{2N-1-2j}.
\end{aligned}$$

But the relation

$$2Na_{j-1}^{(N-1)}(\lambda-1) = 2ja_j^{(N)}(\lambda)$$

shows that

$$\begin{aligned}
p_j^{(N)}(\lambda; n-p) + r_{j-1}^{(N-1)}(\lambda-1) &= (\lambda+n-p-2N)a_j^{(N)}(\lambda) + 2Na_{j-1}^{(N-1)}(\lambda-1) \\
&= (\lambda+n-p-2N+2j)a_j^{(N)}(\lambda) \\
&= -p_j(\lambda; N, p).
\end{aligned}$$

The latter identity implies (4.11).

The analogous proof of (4.12) runs as follows. On the one hand, Theorem 4.1.2 implies

$$\begin{aligned}
D_{2N+1}^{(p \rightarrow p)}(\lambda) &= \sum_{j=1}^N (-1)^{N-j} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (\delta d)^j \iota^* \partial_n^{2N+1-2j} \\
&+ \sum_{j=0}^N (-1)^{N-j} p_j^{(N)}(\lambda; p) (\delta d)^j \iota^* \partial_n^{2N+1-2j} \\
&+ \sum_{j=0}^N (-1)^{N-j} q_j^{(N)}(\lambda-1) (\delta d)^j \delta \iota^* i_{\partial_n} \partial_n^{2N-2j},
\end{aligned}$$

with the coefficients

- $p_j^{(N)}(\lambda; p) = (\lambda+p-2N-1)b_j^{(N)}(\lambda)$,
- $q_j^{(N)}(\lambda-1) = a_j^{(N)}(\lambda-1)$,
- $r_j^{(N-1)}(\lambda-1) = 2Nb_j^{(N-1)}(\lambda-1)$.

On the other hand, Theorem 4.2.2 shows that

$$\begin{aligned}
D_{2N+1}^{(p \rightarrow p-1)}(\lambda) &= \sum_{j=1}^N (-1)^{N-j} \left[p_j(\lambda; N, p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (\delta d)^j \iota^* i_{\partial_n} \partial_n^{2N+1-2j} \\
&+ \sum_{j=0}^N (-1)^{N-j} p_j(\lambda; N, p) (\delta d)^j \iota^* i_{\partial_n} \partial_n^{2N+1-2j} \\
&+ \sum_{j=0}^N (-1)^{N-j-1} q_j^{(N)}(\lambda-1) (\delta d)^j \delta \iota^* \partial_n^{2N-2j},
\end{aligned}$$

with the coefficients

- $p_j(\lambda; N, p) = -(\lambda+n-p-2N+2j-1)b_j^{(N)}(\lambda)$,

- $q_j^{(N)}(\lambda-1) = a_j^{(N)}(\lambda-1)$,
- $r_j^{(N-1)}(\lambda-1) = 2Nb_j^{(N-1)}(\lambda-1)$.

Similarly as above, Lemma 4.3.1 and Lemma 4.3.2 yield

$$\begin{aligned}
& (-1)^{pn+1} \star D_{2N+1}^{(n-p \rightarrow n-p)}(\lambda) \bar{\star} \\
&= \sum_{j=1}^N (-1)^{N-j} \left[p_j^{(N)}(\lambda; n-p) + r_j^{(N-1)}(\lambda-1) \right] (\delta d)^j \iota^* i_{\partial_n} \partial_n^{2N+1-2j} \\
&+ \sum_{j=0}^N (-1)^{N-j} p_j^{(N)}(\lambda; n-p) (\delta d)^j \iota^* i_{\partial_n} \partial_n^{2N+1-2j} \\
&+ \sum_{j=0}^N (-1)^{N-j} q_j^{(N)}(\lambda-1) (\delta d)^j \delta \iota^* \partial_n^{2N-2j}.
\end{aligned}$$

But the relation

$$2Nb_{j-1}^{(N-1)}(\lambda-1) = 2jb_j^{(N)}(\lambda)$$

shows that

$$\begin{aligned}
p_j^{(N)}(\lambda; n-p) + r_{j-1}^{(N-1)}(\lambda-1) &= (\lambda+n-p-2N-1)b_j^{(N)}(\lambda) + 2Nb_{j-1}^{(N-1)}(\lambda-1) \\
&= (\lambda+n-p-2N+2j-1)b_j^{(N)}(\lambda) \\
&= -p_j(\lambda; N, p).
\end{aligned}$$

The latter identity implies (4.12). The proof is complete. \square

Now we recall that the Hodge star operators \star and $\bar{\star}$ are conformally equivariant. More precisely, we have the following result.

Lemma 4.3.4. *We have*

$$d\pi_\lambda^{(n-p)}(X)(\bar{\star}\omega) = \bar{\star} d\pi_{\lambda+n-2p}^{(p)}(X)(\omega) \quad \text{for } \omega \in \Omega^p(\mathbb{R}^n), X \in \mathfrak{g}.$$

Similarly, we have

$$d\pi_\lambda'^{(n-1-p)}(X)(\star\omega) = \star d\pi_{\lambda+n-1-2p}'^{(p)}(X)(\omega) \quad \text{for } \omega \in \Omega^p(\mathbb{R}^{n-1}), X \in \mathfrak{g}'.$$

Proof. The Hodge star operator of (M, g) satisfies the identity

$$\star_{\hat{g}} = e^{(n-2p)\varphi} \star_g, \quad \hat{g} = e^{2\varphi} g$$

on $\Omega^p(M)$. In fact, with obvious notation we find

$$\begin{aligned}
\pi_\lambda^{(n-p)}(\gamma)(\star_g \omega) &= e^{\lambda\Phi_\gamma} \gamma_*(\star_g \omega) \\
&= e^{\lambda\Phi_\gamma} \star_{\gamma_*(g)} \gamma_*(\omega) \\
&= e^{\lambda\Phi_\gamma} \star_{e^{2\Phi_\gamma} g} \gamma_*(\omega) \\
&= e^{\lambda\Phi_\gamma} e^{(n-2p)\Phi_\gamma} \star_g \gamma_*(\omega) \\
&= \star_g e^{(\lambda+n-2p)\Phi_\gamma} \gamma_*(\omega) \\
&= \star_g \pi_{\lambda+n-2p}^{(p)}(\gamma)(\omega)
\end{aligned}$$

for $\omega \in \Omega^p(M)$. The assertions follow by differentiation. \square

Theorem 4.3.3 and Lemma 4.3.4 show that the conformal equivariance of one type of the families of conformal symmetry breaking operators is equivalent to the conformal covariance of the other type. In particular, Theorem 4.2.1 and Theorem 4.2.2 follow from Theorem 4.1.1 and Theorem 4.1.2.

4.4. Operators of the third type. We start with the description of third type operators of even-orders.

Theorem 4.4.1. *Let $N \in \mathbb{N}$. Then the differential operator*

$$D_{2N}^{(0 \rightarrow 1)} : \Omega^0(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}^{n-1})$$

of order $2N$ which is defined by

$$D_{2N}^{(0 \rightarrow 1)} \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} (-1)^{N-j-1} b_j^{(N-1)} (2N-1) d(\delta d)^j \iota_n^* \partial_n^{2N-2j-1}$$

is infinitesimally equivariant in the sense that

$$D_{2N}^{(0 \rightarrow 1)} d\pi_{2N-1,0}^\vee(X) = d\pi_{-1,1}'^\vee(X) D_{2N}^{(0 \rightarrow 1)}, \quad X \in \mathfrak{g}'(\mathbb{R}).$$

Proof. The proof is parallel to that of Theorem 4.1.1. The operator $D_{2N}^{(0 \rightarrow 1)}$ is induced by the singular vector in Theorem 3.4.1/(1). \square

We continue with the formulation of the analogous result for odd-order operators of the third type.

Theorem 4.4.2. *Let $N \in \mathbb{N}$. Then the differential operator*

$$D_{2N+1}^{(0 \rightarrow 1)} : \Omega^0(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}^{n-1})$$

of order $2N+1$ which is defined by

$$D_{2N+1}^{(0 \rightarrow 1)} \stackrel{\text{def}}{=} \sum_{j=0}^N (-1)^{N-j} a_j^{(N)} (2N) d(\delta d)^j \iota_n^* \partial_n^{2N-2j}$$

is infinitesimally equivariant in the sense that

$$D_{2N+1}^{(0 \rightarrow 1)} d\pi_{2N,0}^\vee(X) = d\pi_{-1,1}'^\vee(X) D_{2N+1}^{(0 \rightarrow 1)}, \quad X \in \mathfrak{g}'(\mathbb{R}).$$

Proof. The operator $D_{2N+1}^{(0 \rightarrow 1)}$ is induced by the singular vector in Theorem 3.4.1/(2). \square

Remark 3.4.2 yields

Remark 4.4.3. *For any $N \in \mathbb{N}$, we have*

$$D_N^{(0 \rightarrow 1)} = d\dot{D}_{N-1}^{(0 \rightarrow 0)}(N-1).$$

In addition, there is a first-order operator of the third type.

Theorem 4.4.4. *The differential operator*

$$D_1^{(p \rightarrow p+1)} \stackrel{\text{def}}{=} d\iota^* : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p+1}(\mathbb{R}^{n-1})$$

of first-order is infinitesimally equivariant in the sense that

$$D_1^{(p \rightarrow p+1)} d\pi_{-p,p}^\vee(X) = d\pi_{-(p+1),p+1}'^\vee(X) D_1^{(p \rightarrow p+1)}, \quad X \in \mathfrak{g}'(\mathbb{R}).$$

Proof. The operator $D_1^{(p \rightarrow p+1)}$ is induced by the singular vector in Theorem 3.4.1/(3).¹⁰ \square

4.5. Operators of the fourth type. The operators $D_N^{(0 \rightarrow 1)}$ and $D_1^{(p \rightarrow p+1)}$ of the third type have natural counter parts which map $\Omega^n(\mathbb{R}^n) \rightarrow \Omega^{n-2}(\mathbb{R}^{n-1})$ and $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-2}(\mathbb{R}^{n-1})$, respectively.

We start with the description of the even-order operators of the fourth type.

Theorem 4.5.1. *Let $N \in \mathbb{N}$. Then the differential operator*

$$D_{2N}^{(n \rightarrow n-2)} : \Omega^n(\mathbb{R}^n) \rightarrow \Omega^{n-2}(\mathbb{R}^{n-1})$$

of order $2N$ which is defined by

$$D_{2N}^{(n \rightarrow n-2)} \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} (-1)^{N-j} b_j^{(N-1)} (2N-1) \delta(d\delta)^j \iota^* i_{\partial_n} \partial_n^{2N-2j-1}$$

is infinitesimally equivariant in the sense that

$$D_{2N}^{(n \rightarrow n-2)} d\pi_{2N-1,n}^\vee(X) = d\pi_{-1,n-2}'^\vee(X) D_{2N}^{(n \rightarrow n-2)}, \quad X \in \mathfrak{g}'(\mathbb{R}).$$

Proof. The operator $D_{2N}^{(n \rightarrow n-2)}$ is induced by the singular vector in Theorem 3.5.1/(1). \square

We continue with the formulation of the analogous result for odd-order operators of the fourth type.

Theorem 4.5.2. *Let $N \in \mathbb{N}_0$. Then the differential operator*

$$D_{2N+1}^{(n \rightarrow n-2)} : \Omega^n(\mathbb{R}^n) \rightarrow \Omega^{n-2}(\mathbb{R}^{n-1})$$

of order $2N+1$ which is defined by

$$D_{2N+1}^{(n \rightarrow n-2)} = \sum_{j=0}^N (-1)^{N-j+1} a_j^{(N)} (2N) \delta(d\delta)^j \iota^* i_{\partial_n} \partial_n^{2N-2j}$$

is infinitesimally equivariant in the sense that

$$D_{2N+1}^{(n \rightarrow n-2)} d\pi_{2N,n}^\vee(X) = d\pi_{-1,n-2}'^\vee(X) D_{2N+1}^{(0 \rightarrow 1)}, \quad X \in \mathfrak{g}'(\mathbb{R}).$$

Proof. The operator $D_{2N+1}^{(n \rightarrow n-2)}$ is induced by the singular vector in Theorem 3.5.1/(2). \square

Remark 3.5.2 yields

Remark 4.5.3. *For any $N \in \mathbb{N}$, we have*

$$D_N^{(n \rightarrow n-2)} = \delta \dot{D}_{N-1}^{(n \rightarrow n-1)} (N-1).$$

In addition, there is a first-order operator of the fourth type.

Theorem 4.5.4. *Assume that $p = 2, \dots, n$. The differential operator*

$$D_1^{(p \rightarrow p-2)} \stackrel{\text{def}}{=} \delta \iota^* i_{\partial_n} : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-2}(\mathbb{R}^{n-1})$$

of first-order is infinitesimally equivariant in the sense that

$$D_1^{(p \rightarrow p-2)} d\pi_{p-n,p}^\vee(X) = d\pi_{p-n-1,p-2}'^\vee(X) D_1^{(p \rightarrow p-2)}, \quad X \in \mathfrak{g}'(\mathbb{R}).$$

¹⁰We omit the factor i

Proof. The operator $D_1^{(p \rightarrow p-2)}$ is induced by the singular vector in Theorem 3.5.1/(3).¹¹ \square

4.6. Operators on middle degree forms. Here we briefly describe some special issues related to conformal symmetry breaking operators on middle degree forms. As in Section 4.3, we let \star and $\bar{\star}$ denote the Hodge star operators of the Euclidean metrics on \mathbb{R}^{n-1} and \mathbb{R}^n , respectively. We divide the discussion into three cases.

Case 1a: Let n be odd and $p = \frac{n-1}{2}$.

Let $pr_{\pm} : \Omega^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \rightarrow \Omega_{\pm}^{\frac{n-1}{2}}(\mathbb{R}^{n-1})$ denote the projections onto the eigenspaces of the operator \star . Then the compositions

$$D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda) \stackrel{\text{def}}{=} pr_{\pm} \circ D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2})}(\lambda)$$

are non-trivial and we have

Theorem 4.6.1. *For any $N \in \mathbb{N}$, the families*

$$D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda) : \Omega^{\frac{n-1}{2}}(\mathbb{R}^n) \rightarrow \Omega_{\pm}^{\frac{n-1}{2}}(\mathbb{R}^{n-1})$$

of differential operators of order N are infinitesimally equivariant in the sense that

$$D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda) d\pi_{\lambda, \frac{n-1}{2}}^{\vee}(X) = d\pi'_{\lambda-N, \frac{n-1}{2}}^{\vee}(X) D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda),$$

for all $X \in \mathfrak{g}'(\mathbb{R})$.

Proof. The families are induced by the singular vectors $v_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda)$ (defined in Section 3.6). \square

Case 1b: Let n be odd and $p = \frac{n+1}{2}$.

The compositions

$$D_N^{(\frac{n+1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda) \stackrel{\text{def}}{=} pr_{\pm} \circ D_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2})}(\lambda) \circ \bar{\star}, \quad \lambda \in \mathbb{C}$$

are non-trivial and we have

Theorem 4.6.2. *For any $N \in \mathbb{N}$, the families*

$$D_N^{(\frac{n+1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda) : \Omega^{\frac{n+1}{2}}(\mathbb{R}^n) \rightarrow \Omega_{\pm}^{\frac{n-1}{2}}(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C}$$

of differential operators of order N are infinitesimally equivariant in the sense that

$$D_N^{(\frac{n+1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda) d\pi_{\lambda, \frac{n+1}{2}}^{\vee}(X) = d\pi'_{\lambda-N, \frac{n-1}{2}}^{\vee}(X) D_N^{(\frac{n+1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda)$$

for all $X \in \mathfrak{g}'(\mathbb{R})$.

Proof. The families are induced by the singular vectors $\bar{\star} v_N^{(\frac{n-1}{2} \rightarrow \frac{n-1}{2}), \pm}(\lambda)$ (defined in Section 3.6). \square

Case 2: Let n be even and $p = \frac{n}{2}$.

Let $\Omega_{\pm}^{\frac{n}{2}}(\mathbb{R}^n) \subset \Omega^{\frac{n}{2}}(\mathbb{R}^n)$ be the eigenspaces of the operator $\bar{\star}$. We define

$$D_N^{(\frac{n}{2} \rightarrow \frac{n}{2}), \pm}(\lambda) \stackrel{\text{def}}{=} D_N^{(\frac{n}{2} \rightarrow \frac{n}{2})}(\lambda)|_{\Omega_{\pm}^{\frac{n}{2}}(\mathbb{R}^n)}.$$

These operators are non-trivial and we have

¹¹We omit the factor $-i$

Theorem 4.6.3. *For any $N \in \mathbb{N}$, the families*

$$D_N^{(\frac{n}{2} \rightarrow \frac{n}{2}), \pm}(\lambda) : \Omega_{\pm}^{\frac{n}{2}}(\mathbb{R}^n) \rightarrow \Omega^{\frac{n}{2}}(\mathbb{R}^{n-1}), \quad \lambda \in \mathbb{C}$$

of differential operators of order N are infinitesimally equivariant in the sense that

$$D_N^{(\frac{n}{2} \rightarrow \frac{n}{2}), \pm}(\lambda) d\pi_{\lambda, \frac{n}{2}}^{\vee}(X) = d\pi_{\lambda-N, \frac{n}{2}}^{\vee}(X) D_N^{(\frac{n}{2} \rightarrow \frac{n}{2}), \pm}(\lambda), \quad X \in \mathfrak{g}'(\mathbb{R}).$$

Proof. The families are induced by the singular vectors $v_N^{(\frac{n}{2} \rightarrow \frac{n}{2}), \pm}(\lambda)$ (defined in Section 3.6). \square

We refer to Remark 4.8.7 and Remark 4.8.8 for examples of both types of families in dimension $n = 2$.

4.7. Proof of Theorem 3. The following proof of the classification consists in a combination of previously proved facts.

We recall that conformally covariant differential operators

$$D : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^q(\mathbb{R}^{n-1})$$

of order $N \in \mathbb{N}_0$ which are infinitesimally equivariant in the sense that

$$d\pi_{\eta}^{\prime(q)}(X)D = Dd\pi_{\mu}^{\prime(p)}(X), \quad X \in \mathfrak{g}'(\mathbb{R})$$

or, equivalently,

$$d\pi_{-\eta-q, q}^{\vee}(X)D = Dd\pi_{-\mu-p, p}^{\vee}(X), \quad X \in \mathfrak{g}'(\mathbb{R})$$

(see Remark 2.2.3) correspond to elements of the space

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^q(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{-\eta-q}, \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_{-\mu-p})$$

of singular vectors. Since A acts on $\text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R}))$ by push-forward, it follows that

$$\eta + q = N + \mu + p.$$

Hence it remains to describe the spaces

$$\text{Hom}_{\mathfrak{p}'}(\Lambda^q(\mathfrak{n}'_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda-N}, \text{Pol}_N(\mathfrak{n}^*_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R})) \otimes \mathbb{C}_{\lambda}), \quad \lambda \in \mathbb{C}.$$

Proposition 3.1.3 evaluates the ℓ' -invariance of such homomorphisms. In particular, there are four basic groups of such homomorphisms. The additional $\mathfrak{n}'_+(\mathbb{R})$ -invariance of singular vectors in these groups has been discussed in Sections 3.2–3.5. In fact, we proved that all singular vectors are given by

- first type homomorphisms

$$\xi_n^N P(t) \otimes \text{Id} + \xi_n^{N-1} Q(t) E_n \wedge i_E + \xi_n^{N-2} R(t) \alpha \wedge i_E$$

in the case $q = p$ and general λ (Theorems 3.2.1, 3.2.2),

- second type homomorphisms

$$\xi_n^N P(t) \otimes E_n + \xi_n^{N-1} Q(t) \alpha + \xi_n^{N-2} R(t) E_n \wedge \alpha \wedge i_E$$

in the case $q = p - 1$ and general λ (Theorems 3.3.1, 3.3.2),

- third type homomorphisms

$$\xi_n^{N-1} P(t) i_E$$

for $N \geq 1$ in the case $q = 1$, $p = 0$, $\lambda = N - 1$, and

$$i_E$$

in the case $N = 1$, $q = p + 1$, $\lambda = -p$ (Theorem 3.4.1),

- fourth type homomorphisms

$$\xi_n^{N-1} P(t) E_n \wedge \alpha$$

in the case $q = n - 2$, $p = n$, $\lambda = N - 1$, and

$$E_n \wedge \alpha$$

in the case $N = 1$, $q = p - 2$, $\lambda = -(n - p)$ (Theorem 3.5.1),

- the compositions of these homomorphisms with the operator \star .

The quoted theorems characterize the appropriate polynomials P , Q and R in the variable $t = |\xi'|^2 / \xi_n^2$ with the property that the corresponding homomorphisms are $\mathfrak{n}'_+(\mathbb{R})$ -invariant. In all cases, the singular vectors are unique up a constant multiple. The descriptions of the corresponding differential operators can be found in the following respective theorems:

- Theorem 4.1.1, Theorem 4.1.2,
- Theorem 4.2.1, Theorem 4.2.2,
- Theorem 4.4.1, Theorem 4.4.2, Theorem 4.4.4,
- Theorem 4.5.1, Theorem 4.5.2, Theorem 4.5.4.

The first two sets of results correspond to the first and second type operators in Theorem 3/(1),(2). The operators in the third set of results correspond to the third type operators in Theorem 3/(3),(4). Similarly, the operators in the fourth set of results correspond to the fourth type operators in Theorem 3/(5),(6). This completes the proof.

4.8. Examples. In the present section, we discuss some special cases in more detail. We first display explicit formulas for conformal symmetry breaking operators in low-order cases. Furthermore, we demonstrate the equivariance of the first-order families both by direct calculations (for families acting on one-forms) and by recognizing them as special cases of general conformally covariant families on differential forms. Finally, we describe the relation of the present results to previous discussions of the special case $n = 2$ in [J01] and [KKP14].

Example 4.8.1. *Here we display formulas for conformal symmetry breaking operators of the first and second type up to order 3. In each case, we give formulas in the style of the previous theorems as well as formulas in terms of the operators d , δ , \bar{d} and $\bar{\delta}$. First of all, the zeroth order families are*

$$D_0^{(p \rightarrow p)}(\lambda) = (\lambda + p) \iota^* \quad \text{and} \quad D_0^{(p \rightarrow p-1)}(\lambda) = -(\lambda + n - p) \iota^* i_{\partial_n}.$$

The equivariance of these operators follows from the relations

$$\gamma_*(\iota^*(\omega)) = \iota^*(\gamma_*(\omega)) \quad \text{and} \quad \gamma_*(\iota^* i_{\partial_n}(\omega)) = e^{-\Phi_\gamma} \iota^* i_{\partial_n}(\gamma_*(\omega)), \quad \gamma \in G'.$$

By Theorem 4.1.2 and Theorem 4.2.2 for $N = 0$ and Lemma 4.0.1, the first-order families are

$$D_1^{(p \rightarrow p)}(\lambda) = (\lambda + p - 1) \iota^* \partial_n + d \iota^* i_{\partial_n} = (\lambda + p) d \iota^* i_{\partial_n} + (\lambda + p - 1) \iota^* i_{\partial_n} \bar{d}$$

and

$$D_1^{(p \rightarrow p-1)}(\lambda) = -(\lambda+n-p-1)\iota^* i_{\partial_n} \partial_n - \delta \iota^* = -(\lambda+n-p)\delta \iota^* + (\lambda+n-p-1)\iota^* \bar{\delta}$$

Next, using Theorem 4.1.1 and Theorem 4.2.1 for $N = 1$ and Lemma 4.0.1, the second-order families read

$$\begin{aligned} D_2^{(p \rightarrow p)}(\lambda) &= (\lambda+p-2)\Delta \iota^* + (2\lambda+n-3)(\lambda+p-2)\iota^* \partial_n^2 \\ &\quad + 2(2\lambda+n-3)d\iota^* i_{\partial_n} \partial_n + 2d\delta \iota^* \end{aligned}$$

or, equivalently,

$$\begin{aligned} D_2^{(p \rightarrow p)}(\lambda) &= (2\lambda+n-2) [(\lambda+p)d\delta + (\lambda+p-2)\delta d] \iota^* \\ &\quad - (2\lambda+n-3)\iota^* [(\lambda+p)\bar{d}\bar{\delta} + (\lambda+p-2)\bar{\delta}\bar{d}] \end{aligned}$$

and

$$\begin{aligned} D_2^{(p \rightarrow p-1)}(\lambda) &= -(\lambda+n-p)\Delta \iota^* i_{\partial_n} - (2\lambda+n-3)(\lambda+n-p-2)\iota^* i_{\partial_n} \partial_n^2 \\ &\quad - 2(2\lambda+n-3)\delta \iota^* \partial_n + 2d\delta \iota^* i_{\partial_n} \end{aligned}$$

or, equivalently,

$$\begin{aligned} D_2^{(p \rightarrow p-1)}(\lambda) &= -(2\lambda+n-2) [(\lambda+n-p)\delta d + (\lambda+n-p-2)d\delta] \iota^* i_{\partial_n} \\ &\quad + (2\lambda+n-3)\iota^* i_{\partial_n} [(\lambda+n-p)\bar{\delta}\bar{d} + (\lambda+n-p-2)\bar{d}\bar{\delta}]. \end{aligned}$$

Finally, by Theorem 4.1.2 and Theorem 4.2.2 for $N = 1$ and Lemma 4.0.1, the third-order families are given by

$$\begin{aligned} D_3^{(p \rightarrow p)}(\lambda) &= (\lambda+p-3)\Delta \iota^* \partial_n + \Delta d\iota^* i_{\partial_n} + 2d\delta \iota^* \partial_n \\ &\quad + \frac{1}{3}(2\lambda+n-5)(\lambda+p-3)\iota^* \partial_n^3 + (2\lambda+n-5)d\iota^* i_{\partial_n} \partial_n^2 \end{aligned}$$

or, equivalently,

$$\begin{aligned} D_3^{(p \rightarrow p)}(\lambda) &= \frac{1}{3}(2\lambda+n-2)(\lambda+p)d\delta d\iota^* i_{\partial_n} - \frac{1}{3}(2\lambda+n-5)(\lambda+p-3)\iota^* i_{\partial_n} \bar{d}\bar{\delta}\bar{d} \\ &\quad + \frac{1}{3}(2\lambda+n-2)(\lambda+p-3)\delta d\iota^* i_{\partial_n} \bar{d} - \frac{1}{3}(2\lambda+n-5)(\lambda+p)d\iota^* i_{\partial_n} \bar{d}\bar{\delta} \\ &\quad + \frac{1}{3}[2(\lambda+p-3)(2\lambda+n-2) + 3(\lambda+n-p)]d\delta \iota^* i_{\partial_n} \bar{d} \end{aligned}$$

and

$$\begin{aligned} D_3^{(p \rightarrow p-1)}(\lambda) &= -(\lambda+n-p-1)\Delta \iota^* i_{\partial_n} \partial_n - \Delta \delta \iota^* + 2d\delta \iota^* i_{\partial_n} \partial_n \\ &\quad - \frac{1}{3}(2\lambda+n-5)(\lambda+n-p-3)\iota^* i_{\partial_n} \partial_n^3 - (2\lambda+n-5)\delta \iota^* \partial_n^2 \end{aligned}$$

or, equivalently,

$$\begin{aligned} D_3^{(p \rightarrow p-1)}(\lambda) &= -\frac{1}{3}(2\lambda+n-2)(\lambda+n-p)\delta d\delta \iota^* - \frac{1}{3}(2\lambda+n-5)(\lambda+n-p-3)\iota^* \bar{\delta}\bar{d}\bar{\delta} \\ &\quad + \frac{1}{3}(2\lambda+n-2)(\lambda+n-p-3)d\delta \iota^* \bar{\delta} + \frac{1}{3}(2\lambda+n-5)(\lambda+n-p)\delta \iota^* \bar{\delta}\bar{d} \\ &\quad + \frac{1}{3}[2(\lambda+n-p-3)(2\lambda+n-2) + 3(\lambda+p)]\delta d\iota^* \bar{\delta}. \end{aligned}$$

The respective second forms of the conformal symmetry breaking operators of the first and second type are special cases of the geometric formulas in Section 5. Note that the displayed formulas easily confirm the Hodge conjugation of both types of families.

Example 4.8.2. *For the conformal symmetry breaking operators of the third and fourth type of order $N \leq 4$, we find*

$$\begin{aligned} D_1^{(0 \rightarrow 1)} &= d\iota^* = d\dot{D}_0^{(0 \rightarrow 0)}(0), \\ D_2^{(0 \rightarrow 1)} &= d\iota^* \partial_n = d\dot{D}_1^{(0 \rightarrow 0)}(1), \\ D_3^{(0 \rightarrow 1)} &= d\delta d\iota^* + (n+1)d\iota^* \partial_n^2 = d\dot{D}_2^{(0 \rightarrow 0)}(2), \\ D_4^{(0 \rightarrow 1)} &= \frac{n+1}{3} d\iota^* \partial_n^3 + d\delta d\iota^* \partial_n = d\dot{D}_3^{(0 \rightarrow 0)}(3) \end{aligned}$$

and

$$\begin{aligned} D_1^{(n \rightarrow n-2)} &= -\delta \iota^* i_{\partial_n} = \delta \dot{D}_0^{(n \rightarrow n-1)}(0), \\ D_2^{(n \rightarrow n-2)} &= -\delta \iota^* i_{\partial_n} \partial_n = \delta \dot{D}_1^{(n \rightarrow n-1)}(1), \\ D_3^{(n \rightarrow n-2)} &= -(n+1)\delta \iota^* i_{\partial_n} \partial_n^2 - \delta d\delta \iota^* i_{\partial_n} = \delta \dot{D}_2^{(n \rightarrow n-1)}(2), \\ D_4^{(n \rightarrow n-2)} &= -\frac{n+1}{3} \delta i_{\partial_n} \partial_n^3 - \delta d\delta i_{\partial_n} \partial_n = \delta \dot{D}_3^{(n \rightarrow n-1)}(3). \end{aligned}$$

The displayed formulas easily confirm the Hodge conjugation of both types of operators.

Next, we confirm the equivariance of the first-order families of both types by only using direct arguments.

Remark 4.8.3. *We directly demonstrate the infinitesimal equivariance of the first-order family*

$$D_1^{(1 \rightarrow 1)}(\lambda) = \lambda \iota^* \partial_n + d\iota^* i_{\partial_n} : \Omega^1(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}^{n-1})$$

of the first type (see Remark 4.8.1). Let $1 \leq j \leq n-1$. We use $\Omega^1(\mathbb{R}^n) \simeq C^\infty(\mathbb{R}^n) \otimes \Lambda^1(\mathbb{R}^n)^*$ and determine the action of E_j^+ using the formula (2.29), where we identify $(E_j^\pm)^* \simeq dx^j$. The sum in the formula (2.29) acts on the one-form $\omega = f_l dx^l$ by

$$\sum_{k=1}^n x_k (dx^j \delta_{kl} - dx^k \delta_{jl})(f_l).$$

The latter sum reduces on tangential differential forms to a summation over $k = 1, \dots, n-1$, while on normal differential forms the summation is taken over $k = 1, \dots, n$. Thus, for tangential one-forms $\omega = f_l dx^l$, $l = 1, \dots, n-1$, we obtain

$$\begin{aligned} & D_1^{(1 \rightarrow 1)}(\lambda) d\pi_{\lambda,1}^\vee(E_j^+)(\omega) \\ &= D_1^{(1 \rightarrow 1)}(\lambda) \left(-\frac{1}{2} \sum_{k=1}^n x_k^2 \partial_j dx^k + x_j (-\lambda + \sum_{k=1}^n x_k \partial_k) dx^l + \sum_{k=1}^{n-1} x_k (dx^j \delta_{kl} - dx^k \delta_{jl}) \right) (f_l) \\ &= -\frac{\lambda}{2} \sum_{k=1}^{n-1} x_k^2 \partial_n \partial_j f_l dx^l - \lambda^2 x_j \partial_n f_l dx^l + \lambda x_j \sum_{k=1}^{n-1} x_k \partial_n \partial_k f_l dx^l + \lambda x_j \partial_n f_l dx^l \\ &\quad - \lambda \sum_{k=1}^{n-1} x_k \partial_n f_l dx^k \delta_{jl} + \lambda x_l \partial_n f_l dx^j. \end{aligned}$$

In order to simplify the formulas, we wrote here f_l instead of $\iota^*(f_l)$. But an analogous calculation shows that the latter sum coincides with

$$d\pi'_{\lambda-1,1}(E_j^+)D_1^{(1\rightarrow 1)}(\lambda)(\omega) = \lambda d\pi'_{\lambda-1,1}(E_j^+)(\partial_n f_l dx^l).$$

Similarly, for normal forms $\omega = f_n dx^n$, we obtain

$$\begin{aligned} & D_1^{(1\rightarrow 1)}(\lambda)d\pi'_{\lambda,1}(E_j^+)(\omega) \\ &= D_1^{(1\rightarrow 1)}(\lambda) \left(-\frac{1}{2} \sum_{k=1}^n x_k^2 \partial_j dx^n + x_j \left(-\lambda + \sum_{k=1}^n x_k \partial_k \right) dx^n + x_n dx^j \right) (f_n) \\ &= -\frac{1}{2} \sum_{k,l=1}^{n-1} x_k^2 \partial_j \partial_l f_n dx^l + x_j \sum_{k,l=1}^{n-1} x_k \partial_k \partial_l f_n dx^l \\ &\quad - (\lambda - 1) x_j \sum_{k=1}^{n-1} \partial_k f_n dx^k - \sum_{k=1}^{n-1} x_k \partial_j f_n dx^k + \sum_{k=1}^{n-1} x_k \partial_k f_n dx^j. \end{aligned}$$

But an analogous calculation shows that the latter sum coincides with

$$d\pi'_{\lambda-1,1}(E_j^+)D_1^{(1\rightarrow 1)}(\lambda)(\omega) = d\pi'_{\lambda-1,1}(E_j^+)(d\iota^* f_n).$$

Thus we have proved that

$$D_1^{(1\rightarrow 1)}(\lambda)d\pi'_{\lambda,1}(E_j^+) = d\pi'_{\lambda-1,1}(E_j^+)D_1^{(1\rightarrow 1)}(\lambda), \quad 1 \leq j \leq n-1.$$

Now $(d\pi_{\lambda,1})^\vee(E_j^-) = \partial/\partial x_j$ and $SO(n-1, \mathbb{R})$ both commute with $D_1^{(1\rightarrow 1)}(\lambda)$. The assertion is obvious for $X = E$. This completes the proof of the equivariance for $\mathfrak{g}'(\mathbb{R})$.

Remark 4.8.4. Here we sketch an analogous direct proof of the infinitesimal equivariance of the first-order family

$$D_1^{(1\rightarrow 0)}(\lambda) = -(\lambda+n-2)\iota^* i_{\partial_n} \partial_n - \delta \iota^* : \Omega^1(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$$

of the second type. We shall use the same conventions as in Remark 4.8.3. Again, the non-trivial part of the assertion concerns the actions of $X = E_j^+$. On tangential forms $\omega = f_l x^l$, $l = 1, \dots, n-1$, we obtain

$$\begin{aligned} & D_1^{(1\rightarrow 0)}(\lambda)d\pi'_{\lambda,1}(E_j^+)(\omega) \\ &= D_1^{(1\rightarrow 0)}(\lambda) \left(-\frac{1}{2} \sum_{k=1}^n x_k^2 \partial_j dx^l + x_j \left(-\lambda + \sum_{k=1}^n x_k \partial_k \right) dx^l + \sum_{k=1}^n x_k (dx^j \delta_{kl} - dx^k \delta_{jl}) \right) (f_l) \\ &= -\frac{1}{2} \sum_{k=1}^{n-1} x_k^2 \partial_l \partial_j f_l - (\lambda - 1) x_j \partial_l f_l + x_j \sum_{k=1}^{n-1} x_k \partial_l \partial_k f_l. \end{aligned}$$

An analogous calculation shows that the latter sum coincides with

$$d\pi'_{\lambda-1,0}(E_j^+)D_1^{(1\rightarrow 0)}(\lambda)(\omega) = d\pi'_{\lambda-1,0}(E_j^+)(\partial_l f_l).$$

Similarly, for normal forms $\omega = f_n dx^n$, we obtain

$$\begin{aligned} & D_1^{(1\rightarrow 0)}(\lambda)d\pi'_{\lambda,1}(E_j^+)(\omega) \\ &= D_1^{(1\rightarrow 0)}(\lambda) \left(-\frac{1}{2} \sum_{k=1}^n x_k^2 \partial_j dx^n - \lambda x_j dx^n + x_j \sum_{k=1}^n x_k \partial_k dx^n + x_n dx^j \right) (f_n) \end{aligned}$$

$$\begin{aligned}
&= (\lambda+n-2) \frac{1}{2} \sum_{k=1}^{n-1} x_k^2 \partial_n \partial_j f_n - (\lambda+n-2) x_j \sum_{k=1}^{n-1} x_k \partial_n \partial_k f_n \\
&\quad + \lambda(\lambda+n-2) x_j \partial_n f_n - (\lambda+n-2) x_j \partial_n f_n.
\end{aligned}$$

An analogous calculation shows that the latter sum coincides with

$$d\pi'_{\lambda-1,0}{}^\vee(E_j^+) D_1^{(1 \rightarrow 0)}(\lambda)(\omega) = -(\lambda+n-2) d\pi'_{\lambda-1,0}{}^\vee(E_j^+)(\partial_n f_n).$$

Thus we have proved that

$$D_1^{(1 \rightarrow 0)}(\lambda) d\pi'_{\lambda,1}{}^\vee(E_j^+) = d\pi'_{\lambda-1,0}{}^\vee(E_j^+) D_1^{(1 \rightarrow 0)}(\lambda), \quad 1 \leq j \leq n-1.$$

This completes the proof of the equivariance for $\mathfrak{g}'(\mathbb{R})$.

The first-order families $D_1^{(1 \rightarrow 1)}(\lambda)$ and $D_1^{(1 \rightarrow 0)}(\lambda)$ studied in Remark 4.8.3 and Remark 4.8.4 are special cases of conformally covariant families $\Omega^p(X) \rightarrow \Omega^p(M)$ and $\Omega^p(X) \rightarrow \Omega^{p-1}(M)$ which are naturally associated to any codimension one embedding $\iota : M \hookrightarrow X$ of Riemannian manifolds. More precisely, we have the following results.

Lemma 4.8.5. *For any hypersurface $\iota : M \hookrightarrow X$ and any metric g on X , the family*

$$D_1^{(p \rightarrow p)}(g; \lambda) \stackrel{\text{def}}{=} \lambda \iota^* i_N d + (\lambda+1) d\iota^* i_N - \lambda(\lambda+1) H \iota^* : \Omega^p(X) \rightarrow \Omega^p(M)$$

is conformally covariant in the sense that

$$e^{-\lambda \iota^*(\varphi)} D_1^{(p \rightarrow p)}(e^{2\varphi} g; \lambda) = D_1^{(p \rightarrow p)}(g; \lambda) e^{-(\lambda+1)\varphi}$$

for all $\varphi \in C^\infty(X)$. Here N and H denote the unit normal vector field of M and the mean curvature, respectively.

Proof. Using $\hat{N} = e^{-\varphi} N$ and the transformation property

$$e^\varphi \hat{H} = H + \langle d\varphi, N \rangle, \tag{4.13}$$

we find

$$\begin{aligned}
&e^{-\lambda \iota^*(\varphi)} D_1^{(p \rightarrow p)}(e^{2\varphi} g; \lambda) (e^{(\lambda+1)\varphi} \omega) \\
&\quad = \lambda \iota^* (i_N d\omega + (\lambda+1) i_N (d\varphi \wedge \omega)) + (\lambda+1) e^{-\lambda \iota^*(\varphi)} d\iota^* e^{-\varphi} i_N (e^{(\lambda+1)\varphi} \omega) \\
&\quad \quad - \lambda(\lambda+1) (H + \langle d\varphi, N \rangle) \iota^* \omega
\end{aligned}$$

for any $\omega \in \Omega^p(M)$. A simple computation shows that the latter sum equals

$$\lambda \iota^* i_N d\omega + (\lambda+1) d\iota^* i_N \omega - \lambda(\lambda+1) H \iota^* \omega.$$

The proof is complete. \square

The conformally covariant family $D_1^{(p \rightarrow p)}(g; \lambda)$ (in Lemma 4.8.5) clearly generalizes the family $D_1^{(p \rightarrow p)}(\lambda)$ (displayed in Example 4.8.1) (up to the shift $\lambda \mapsto \lambda - p + 1$). Therefore, Lemma 4.8.5 yields an alternative proof of the equivariance of $D_1^{(p \rightarrow p)}(\lambda)$.

Lemma 4.8.6. *For any hypersurface $\iota : M^{n-1} \hookrightarrow X^n$ and any metric g on X , the family*

$$\begin{aligned}
D_1^{(p \rightarrow p-1)}(g; \lambda) &\stackrel{\text{def}}{=} (n-2p+\lambda+1) \iota^* \delta_g - (n-2p+\lambda+2) \delta_{\iota^* g} \iota^* \\
&\quad + (n-2p+\lambda+1)(n-2p+\lambda+2) i_{\mathcal{H}} : \Omega^p(X) \rightarrow \Omega^{p-1}(M)
\end{aligned}$$

is conformally covariant in the sense that

$$e^{-\lambda\iota^*(\varphi)} D_1^{(p \rightarrow p-1)}(e^{2\varphi} g; \lambda) = D_1^{(p \rightarrow p-1)}(g; \lambda) e^{-(\lambda+2)\varphi}$$

for all $\varphi \in C^\infty(X)$. Here $\mathcal{H} = HN$ denotes the mean-curvature vector.

Proof. The conformal transformation law

$$e^{-(a-2)\varphi} \circ \hat{\delta} \circ e^{a\varphi} = \delta - (n-2p+a)i_{\text{grad}(\varphi)}$$

for p -forms on a manifold of dimension n implies

$$e^{-\lambda\varphi} \circ \delta_{\hat{g}} = \delta_g \circ e^{-(\lambda+2)\varphi} - (n-2p+\lambda+2)i_{\text{grad}_g(\varphi)} \circ e^{-(\lambda+2)\varphi}$$

for $\varphi \in C^\infty(X)$ on p -forms on X , and

$$e^{-\lambda\psi} \circ \delta_{\hat{g}} = \delta_g \circ e^{-(\lambda+2)\psi} - (n-2p+\lambda+1)i_{\text{grad}_g(\psi)} \circ e^{-(\lambda+2)\psi}$$

for $\psi \in C^\infty(M)$ on p -forms on M . It follows that

$$e^{-\lambda\iota^*(\varphi)} \circ ((n-2p+\lambda+1)\iota^*\delta_{\hat{g}} - (n-2p+\lambda+2)\delta_{\iota^*\hat{g}}\iota^*)$$

differs from

$$((n-2p+\lambda+1)\iota^*\delta_{\hat{g}} - (n-2p+\lambda+2)\delta_{\iota^*\hat{g}}\iota^*) \circ e^{(\lambda-2)\varphi}$$

by

$$-(n-2p+\lambda+1)(n-2p+\lambda+2)\iota^*i_{N(\varphi)} \circ e^{(\lambda-2)\varphi},$$

where $N(\varphi) = \langle d\varphi, N \rangle N$. But the conformal transformation law (4.13) implies that

$$e^{-\lambda\iota^*(\varphi)} \circ i_{\mathcal{H}} = (i_{\mathcal{H}} + i_{N(\varphi)}) \circ e^{-(\lambda+2)\varphi}.$$

This completes the proof. \square

The conformally covariant family $D_1^{(p \rightarrow p-1)}(g; \lambda)$ (in Lemma 4.8.6) clearly generalizes the family $D_1^{(p \rightarrow p-1)}(\lambda)$ (displayed in Example 4.8.1) (up to the shift $\lambda \mapsto \lambda - p + 2$). Therefore, Lemma 4.8.6 yields an alternative proof of the equivariance of $D_1^{(p \rightarrow p-1)}(\lambda)$.

The following remark illustrates Theorem 4.6.3.

Remark 4.8.7. In dimension $n = 2$, the conformal symmetry breaking operators $D_1^{(1 \rightarrow 1)}(\lambda)$ and $D_2^{(1 \rightarrow 1)}(\lambda)$ of the first type appeared already in [J01, Equations (8.200)–(8.202)] as low-order special cases of so-called relative differential intertwining operators which are induced by homomorphisms of generalized Verma modules. In more details, the relation is the following. In terms of the coordinates x, y on \mathbb{R}^2 with the normal variable y of the subspace \mathbb{R}^1 , we have the formulas

$$D_1^{(1 \rightarrow 1)}(\lambda) = -\lambda\iota^*\partial_y + d\iota^*i_{\partial_y}$$

and

$$D_2^{(1 \rightarrow 1)}(\lambda) = -(\lambda+1)\partial_x^2\iota^* + (2\lambda-1)(\lambda-1)\iota^*\partial_y^2 + 2(2\lambda-1)d\iota^*i_{\partial_y}\partial_y$$

(see Example 4.8.1). By restriction to $\omega = f(dx \pm idy) \in \Omega^1(\mathbb{R}^2)$, we find

$$D_1^{(1 \rightarrow 1)}(\lambda)(\omega) = \mp i\lambda\iota^*(\partial_y f)dx + d\iota^*(f)$$

and

$$D_2^{(1 \rightarrow 1)}(\lambda)(\omega) = -(\lambda+1)\iota^*(\partial_x^2 f)dx + (2\lambda-1)(\lambda-1)\iota^*(\partial_y^2 f)dx \pm 2i(2\lambda-1)d\iota^*(\partial_y f).$$

By Lemma 2.2.2, the subspaces $\{f(dx \pm dy)\}$ are the eigenspaces of the Hodge star operator on \mathbb{R}^2 . Using respective shifts of λ by 1 and 2, we obtain the operators

$$f(dx \pm idy) \mapsto \mp i(\lambda + 1)\iota^*(\partial_y f)dx + d\iota^*(f)$$

and

$$f(dx \pm idy) \mapsto -(\lambda + 3)\iota^*(\partial_x^2 f)dx + (2\lambda + 3)(\lambda + 1)\iota^*(\partial_y^2 f)dx \pm 2i(2\lambda + 3)d\iota^*(\partial_y f).$$

These are the operators displayed in [J01]. This shows that the family

$$D_2^{(1 \rightarrow 1)}(\lambda) : \Omega^1(\mathbb{R}^2) \rightarrow \Omega^1(\mathbb{R})$$

of the first type restricts to two different families $\Omega_{\pm}^1(\mathbb{R}^2) \rightarrow \Omega^1(\mathbb{R})$.

Remark 4.8.8. The special case $\Omega^1(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R})$ of the second type families of order $N \in \mathbb{N}$ was analyzed in [KKP14]. We show that their result is a special case of ours. We use the expansions

$$C_m^\alpha(t) = \sum_{k=0}^{[m/2]} (-1)^k \frac{\Gamma(m - k + \alpha)}{\Gamma(\alpha)(m - 2k)!k!} (2t)^{m-2k}, \quad m \in \mathbb{N}_0$$

(see (6.58)) to define homogeneous differential operators

$$C_m^\alpha \stackrel{\text{def}}{=} (\mathrm{i}\partial_x)^m C_m^\alpha \left(\frac{\partial_y}{\mathrm{i}\partial_x} \right).$$

In particular, we find

$$\begin{aligned} C_0^\alpha &= \mathrm{Id}, \\ C_1^\alpha &= 2\lambda\partial_y, \\ C_2^\alpha &= \lambda(\partial_x^2 + 2(\lambda + 1)\partial_y^2), \\ C_3^\alpha &= \frac{2}{3}\lambda(\lambda + 1)(3\partial_x^2\partial_y + 2(\lambda + 2)\partial_y^3). \end{aligned}$$

Furthermore, we define the operators

$$D_m^1(\lambda) \stackrel{\text{def}}{=} m(2\lambda + m - 1)\partial_x(\mathrm{i}\partial_x)^{m-1}C_{m-1}^{\lambda+\frac{1}{2}} \left(\frac{\partial_y}{\mathrm{i}\partial_x} \right) \quad (4.14)$$

and

$$\begin{aligned} D_m^2(\lambda) &\stackrel{\text{def}}{=} (2\lambda^2 + 2(m-1)\lambda + m(m-1))\partial_y(\mathrm{i}\partial_x)^{m-1}C_{m-1}^{\lambda+\frac{1}{2}} \left(\frac{\partial_y}{\mathrm{i}\partial_x} \right) \\ &\quad + (\lambda-1)(2\lambda+1)(\partial_x^2 + \partial_y^2)(\mathrm{i}\partial_x)^{m-2}C_{m-2}^{\lambda+\frac{3}{2}} \left(\frac{\partial_y}{\mathrm{i}\partial_x} \right). \end{aligned} \quad (4.15)$$

Let $\iota : \mathbb{R} \hookrightarrow \mathbb{R}^2$ be defined by $x \mapsto (x, 0)$.

Proposition 4.8.9. Let $N \in \mathbb{N}$. The operators $D_N^1(\lambda)$ and $D_N^2(\lambda)$ are related to the conformal symmetry breaking operators $D_N^{(1 \rightarrow 0)}(\lambda)$ of the second type through the identities

$$\iota^*(D_{2N}^1(\lambda)(f) + D_{2N}^2(\lambda)(g)) = (-1)^N \frac{2(\lambda + \frac{1}{2})_N}{(N-1)!} D_{2N}^{(1 \rightarrow 0)}(-\lambda)(f dx + g dy) \quad (4.16)$$

and

$$\iota^*(D_{2N+1}^1(\lambda)(f) + D_{2N+1}^2(\lambda)(g)) = (-1)^N \frac{2(2N+1)(\lambda + \frac{1}{2})_N(\lambda + N)}{N!} D_{2N+1}^{(1 \rightarrow 0)}(-\lambda)(f dx + g dy) \quad (4.17)$$

for $f, g \in C^\infty(\mathbb{R}^2)$.

Proof. The proof rests on the identities

$$\begin{aligned} C_{2N}^{\lambda + \frac{1}{2}}(z) &= \frac{(\lambda + \frac{1}{2})_N}{N!} \sum_{j=0}^N (-1)^j a_j^{(N)} (-\lambda - 1) z^{2N-2j}, \\ C_{2N+1}^{\lambda + \frac{1}{2}}(z) &= \frac{2(\lambda + \frac{1}{2})_{N+1}}{N!} \sum_{j=0}^N (-1)^j b_j^{(N)} (-\lambda - 1) z^{2N-2j+1} \end{aligned} \quad (4.18)$$

(see the Appendix). By formula (4.7) in Theorem 4.2.1 the even-order family $D_{2N}^{(1 \rightarrow 0)}(\lambda)$ acts on $\omega = f dx + g dy \in \Omega^1(\mathbb{R}^2)$ by

$$D_{2N}^{(1 \rightarrow 0)}(\lambda)(\omega) = \iota^*(D_{2N}^{(1 \rightarrow 0),1}(\lambda)(f) + D_{2N}^{(1 \rightarrow 0),2}(\lambda)(g)),$$

where

$$\begin{aligned} D_{2N}^{(1 \rightarrow 0),1}(\lambda) &\stackrel{\text{def}}{=} (-1)^N (i\partial_x)^{2N-1} \partial_x \sum_{j=0}^{N-1} (-1)^j q_j^{(N-1)} (\lambda - 1) \left(\frac{\partial_y}{i\partial_x} \right)^{2N-2j-1}, \\ D_{2N}^{(1 \rightarrow 0),2}(\lambda) &\stackrel{\text{def}}{=} (-1)^N (i\partial_x)^{2N} \sum_{j=0}^N (-1)^j p_j(\lambda; N, 1) \left(\frac{\partial_y}{i\partial_x} \right)^{2N-2j} \end{aligned}$$

and

$$\begin{aligned} q_j^{(N-1)}(\lambda - 1) &= -2N(2\lambda - 2N + 1) b_j^{(N-1)}(\lambda - 1), \\ p_j(\lambda; N, 1) &= -(\lambda - 2N + 2j + 1) a_j^{(N)}(\lambda). \end{aligned}$$

It follows that the claim (4.16) is equivalent to the relations

$$D_{2N}^1(\lambda) = (-1)^N \frac{2(\lambda + \frac{1}{2})_N}{(N-1)!} D_{2N}^{(1 \rightarrow 0),1}(-\lambda), \quad (4.19)$$

$$D_{2N}^2(\lambda) = (-1)^N \frac{2(\lambda + \frac{1}{2})_N}{(N-1)!} D_{2N}^{(1 \rightarrow 0),2}(-\lambda). \quad (4.20)$$

Now (4.19) directly follows from the definition (4.14). We proceed with the proof of (4.20). By definition and (4.18), the left-hand side equals

$$\begin{aligned} D_{2N}^2(\lambda) &= \frac{2(\lambda + \frac{1}{2})_N}{(N-1)!} \left[\sum_{j=0}^N (-1)^j \left[(2N(2\lambda + 2N - 1) + 2\lambda(\lambda - 1)) b_j^{(N-1)}(-\lambda - 1) \right. \right. \\ &\quad \left. \left. + (\lambda - 1) a_{j-1}^{(N-1)}(-\lambda - 2) + (\lambda - 1) a_j^{(N-1)}(-\lambda - 2) \right] (i\partial_x)^{2j} \partial_y^{2N-2j} \right]. \end{aligned}$$

Here we have set $a_{-1}^{(N-1)}(\lambda) \stackrel{\text{def}}{=} 0$, $a_N^{(N-1)}(\lambda) \stackrel{\text{def}}{=} 0$ and $b_N^{(N-1)}(\lambda) \stackrel{\text{def}}{=} 0$. Now an elementary computation shows that

$$2\lambda b_j^{(N-1)}(-\lambda-1) + a_{j-1}^{(N-1)}(-\lambda-2) + a_j^{(N-1)}(-\lambda-2) = a_j^{(N)}(-\lambda)$$

for $j = 0, \dots, N$. Hence the identity

$$p_j(-\lambda; N, 1) = (\lambda-1)a_j^{(N)}(-\lambda) + 2N(2\lambda+2N-1)b_j^{(N-1)}(-\lambda-1)$$

proves (4.20). Similar arguments can be used to prove (4.17). The proof is complete. \square

The main result of [KKP14] states that, for any $m \in \mathbb{N}_0$, the family

$$\Omega^1(\mathbb{R}^2) \ni \omega = fdx + gdy \mapsto \iota^*(D_m^1(\lambda)(f) + D_m^2(\lambda)(g)) \in C^\infty(\mathbb{R})$$

satisfies the same intertwining relation as $D_m^{(1,0)}(-\lambda)$. Moreover, the authors observed that the compositions of these families with the Hodge star operator on $\Omega^1(\mathbb{R}^2)$ define additional families with the same equivariance. These results follow from Proposition 4.8.9.

5. GEOMETRIC FORMULAS FOR CONFORMAL SYMMETRY BREAKING OPERATORS

In the present section, we apply the results in Section 4 to derive formulas for all types of conformal symmetry breaking operators in terms of the four geometric operators d , δ , \bar{d} , $\bar{\delta}$, the pull-back ι^* and the insertion i_{∂_n} of the normal vector field. We shall refer to these formulas as to geometric formulas for the families. These results generalize the low-order examples ($N \leq 3$) displayed in Example 4.8.1.

5.1. Preparations. We note that Lemma 4.0.1 implies the identities

$$\iota^* \partial_n^{2k} = \sum_{i=0}^k (-1)^i \binom{k}{i} \Delta^{k-i} \iota^* \bar{\Delta}^i, \quad (5.1)$$

$$\iota^* i_{\partial_n} \partial_n^{2k+1} = \sum_{i=0}^k (-1)^i \binom{k}{i} \Delta^{k-i} (\delta \iota^* - \iota^* \bar{\delta}) \bar{\Delta}^i \quad (5.2)$$

and

$$\iota^* i_{\partial_n} \partial_n^{2k} = \sum_{i=0}^k (-1)^i \binom{k}{i} \Delta^{k-i} \iota^* i_{\partial_n} \bar{\Delta}^i, \quad (5.3)$$

$$\iota^* \partial_n^{2k-1} = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \Delta^{k-i-1} (d \iota^* i_{\partial_n} + \iota^* i_{\partial_n} \bar{d}) \bar{\Delta}^i. \quad (5.4)$$

Indeed, Lemma 4.0.1/(3) yields $\partial_n^2 = \Delta - \bar{\Delta}$. Hence $\partial_n^{2k} = \sum_{i=0}^k (-1)^i \binom{k}{i} \Delta^{k-i} \bar{\Delta}^i$. This proves (5.1). Moreover, $i_{\partial_n} \partial_n = \delta - \bar{\delta}$ (Lemma 4.0.1/(2)) gives

$$\begin{aligned} \iota^* i_{\partial_n} \partial_n^{2k+1} &= \iota^* i_{\partial_n} \partial_n \partial_n^{2k} = (\delta \iota^* - \iota^* \bar{\delta}) \sum_{i=0}^k (-1)^i \binom{k}{i} \Delta^{k-i} \bar{\Delta}^i \\ &= \sum_{i=0}^k (-1)^i \binom{k}{i} \Delta^{k-i} (\delta \iota^* - \iota^* \bar{\delta}) \bar{\Delta}^i. \end{aligned}$$

This proves (5.2). Similar arguments prove (5.3) and (5.4).

Furthermore, we introduce the coefficients

$$\alpha_i^{(N)}(\lambda) \stackrel{\text{def}}{=} (-1)^i 2^N \frac{N!}{(2N)!} \binom{N}{i} \prod_{k=i+1}^N (2\lambda+n-2k) \prod_{k=1}^i (2\lambda+n-2k-2N+1) \quad (5.5)$$

and

$$\beta_i^{(N)}(\lambda) \stackrel{\text{def}}{=} (-1)^i 2^N \frac{N!}{(2N+1)!} \binom{N}{i} \prod_{k=i+1}^N (2\lambda+n-2k) \prod_{k=1}^i (2\lambda+n-2k-2N-1) \quad (5.6)$$

for $N \in \mathbb{N}_0$ and $i = 0, \dots, N$. By convention, empty products are defined as 1.

The following observation will be useful to identify certain series as hypergeometric functions. A series $\sum_{n \geq 0} c_n$ is a hypergeometric function

$$c_0 {}_2F_1(a, b; c; x) = c_0 \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

iff

$$\frac{c_{n+1}}{c_n} = \frac{(n+a)(n+b)}{n+c} \frac{x}{n+1}.$$

We also recall the Zhu-Vandermonde formula

$${}_2F_1(-n, b; c; 1) = \sum_{j=0}^n \frac{(-n)_j (b)_j}{(c)_j} \frac{1}{j!} = \frac{(c-b)_n}{(c)_n}, \quad n \in \mathbb{N}_0. \quad (5.7)$$

Remark 5.1.1. We identify the generating polynomials for the coefficients $\alpha_i^{(N)}(\lambda)$ and $\beta_i^{(N)}(\lambda)$ as Jacobi polynomials. Indeed, the relations

$$\binom{N}{i} = (-1)^i \frac{(-N)_i}{i!},$$

$$\prod_{k=i+1}^N (2\lambda+n-2k) = 2^{N-i} (\lambda + \frac{n}{2} - N)_{N-i} = (-2)^{N-i} \frac{(-\lambda - \frac{n}{2} + 1)_N}{(-\lambda - \frac{n}{2} + 1)_i},$$

$$\prod_{k=1}^i (2\lambda+n-2k-2N+1) = (-2)^i (N + \frac{1}{2} - \lambda - \frac{n}{2})_i$$

imply that

$$\sum_{i=0}^N \alpha_i^{(N)}(\lambda) t^i = 4^N \frac{N!}{(2N)!} (\lambda + \frac{n}{2} - N)_N {}_2F_1 \left[\begin{matrix} -N, N + \frac{1}{2} - \lambda - \frac{n}{2} \\ 1 - \lambda - \frac{n}{2} \end{matrix}; t \right].$$

The right-hand side is proportional to $P_N^{(-\lambda - \frac{n}{2}, -\frac{1}{2})}(1-2t)$. Similarly, we find

$$\sum_{i=0}^N \beta_i^{(N)}(\lambda) t^i = 4^N \frac{N!}{(2N+1)!} (\lambda + \frac{n}{2} - N)_N {}_2F_1 \left[\begin{matrix} -N, N + \frac{3}{2} - \lambda - \frac{n}{2} \\ 1 - \lambda - \frac{n}{2} \end{matrix}; t \right].$$

The right-hand side is proportional to $P_N^{(-\lambda - \frac{n}{2}, \frac{1}{2})}(1-2t)$. For the definition of Jacobi polynomials $P_N^{(\alpha, \beta)}(t)$ we refer to the Appendix.

The following relations will be useful later on.

Lemma 5.1.2. *We have*

$$\begin{aligned} \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} a_j^{(N)}(\lambda) &= \alpha_i^{(N)}(\lambda), \\ \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} b_j^{(N)}(\lambda) &= \beta_i^{(N)}(\lambda) \end{aligned}$$

for all $i = 0, \dots, N$.

Proof. First, we note that Gegenbauer coefficients can be written in the form

$$a_j^{(N)}(\lambda) = (-4)^{N-j} \frac{N!}{j!(2N-2j)!} \frac{(\lambda + \frac{n}{2} - 2N + \frac{1}{2})_N}{(\lambda + \frac{n}{2} - 2N + \frac{1}{2})_j} \quad (5.8)$$

and

$$b_j^{(N)}(\lambda) = (-4)^{N-j} \frac{N!}{j!(2N-2j+1)!} \frac{(\lambda + \frac{n}{2} - 2N - \frac{1}{2})_N}{(\lambda + \frac{n}{2} - 2N - \frac{1}{2})_j} \quad (5.9)$$

for $0 \leq j \leq N-1$ and $a_N^{(N)}(\lambda) = b_N^{(N)}(\lambda) = 1$. Now (5.8) implies

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} a_j^{(N)}(\lambda) = \sum_{j=0}^{N-i} c_j$$

with

$$\frac{c_{j+1}}{c_j} = \frac{(-N+i+j)(-N+\frac{1}{2}+j)}{(\lambda + \frac{n}{2} - 2N + \frac{1}{2} + j)} \frac{1}{j+1}.$$

Hence

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} a_j^{(N)}(\lambda) = (-1)^{N-i} \binom{N}{i} a_0^{(N)}(\lambda) {}_2F_1 \left[\begin{matrix} \frac{1}{2} - N, i - N \\ \lambda + \frac{n}{2} - 2N + \frac{1}{2} \end{matrix}; 1 \right].$$

By the Zhu-Vandermonde formula (5.7), we obtain

$$\begin{aligned} \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} a_j^{(N)}(\lambda) &= (-1)^{N-i} \binom{N}{i} (-4)^N \frac{N!}{(2N)!} (\lambda + \frac{n}{2} - 2N + \frac{1}{2})_N \frac{(\lambda + \frac{n}{2} - N)_{N-i}}{(\lambda + \frac{n}{2} - 2N + \frac{1}{2})_{N-i}} \\ &= \alpha_i^{(N)}(\lambda). \end{aligned}$$

Similar arguments using (5.9) prove the second relation. \square

5.2. Even-order families of the first and second type. The following result for even-order families of the first type basically restates Theorem 1 in Section 1.

Theorem 5.2.1. *Assume that $N \in \mathbb{N}$ and $p = 0, \dots, n-1$. The even-order families $D_{2N}^{(p \rightarrow p)}(\lambda)$ of the first type can be written in the form*

$$D_{2N}^{(p \rightarrow p)}(\lambda) = (\lambda + p) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i$$

$$\begin{aligned}
& + \sum_{i=1}^{N-1} (\lambda + p - 2i) \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^* (\bar{\delta} \bar{d})^i \\
& + (\lambda + p - 2N) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (\delta d)^{N-i} \iota^* (\bar{\delta} \bar{d})^i
\end{aligned} \tag{5.10}$$

with the coefficients $\alpha_i^{(N)}(\lambda)$ defined by (5.5).

For $p = 0$, the families in Theorem 5.2.1 reduce to the product of $(\lambda - 2N)$ and the equivariant even-order families studied in [J09].

Proof. In the following, we shall use the conventions $r_{-1}^{(N-1)}(\lambda) = 0$ and $q_{-1}^{(N-1)}(\lambda) = 0$. We start by proving the formula

$$\begin{aligned}
D_{2N}^{(p \rightarrow p)}(\lambda) &= \sum_{i=0}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* (\bar{\delta} \bar{d})^i \\
&+ \sum_{i=0}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} S_j(\lambda; N, p) (\delta d)^{N-i} \iota^* (\bar{\delta} \bar{d})^i \\
&+ \sum_{i=1}^{N-1} \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} T_j(\lambda; N, p) (\delta d)^{N-i} \iota^* (\bar{\delta} \bar{d})^i
\end{aligned} \tag{5.11}$$

with

$$S_j(\lambda; N, p) \stackrel{\text{def}}{=} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda - 1) + q_j^{(N-1)}(\lambda - 1) \right], \quad j = 0, \dots, N$$

and

$$T_j(\lambda; N, p) \stackrel{\text{def}}{=} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda - 1) - q_{j-1}^{(N-1)}(\lambda - 1) \right], \quad j = 0, \dots, N-1.$$

By combining Theorem 4.1.1 with the relations (5.1) and (5.2), we obtain

$$\begin{aligned}
D_{2N}^{(p \rightarrow p)}(\lambda) &= \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda - 1) \right] (\delta d)^{N-i} \iota^* \bar{\Delta}^i \\
&+ \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) ((\delta d)^{N-i} + (\delta d)^{N-i}) \iota^* \bar{\Delta}^i \\
&- p_0^{(N)}(\lambda; p) \iota^* \bar{\Delta}^N \\
&+ \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* \bar{\Delta}^i \\
&+ \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i-1} \binom{N-j}{i} q_{j-1}^{(N-1)}(\lambda - 1) (\delta d)^{N-i} \iota^* \bar{\Delta}^i \\
&+ \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} q_{j-1}^{(N-1)}(\lambda - 1) (\delta d)^{N-i-1} \iota^* (\bar{\delta} \bar{d})^{i+1}
\end{aligned}$$

using $(d\delta)^j \Delta^{N-j-i} = (d\delta)^{N-i}$ if $j \geq 1$. Next, we expand the powers of Laplacians $\bar{\Delta}^i$. We obtain

$$\begin{aligned}
D_{2N}^{(p \rightarrow p)}(\lambda) &= \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} \iota^* (\bar{d}\bar{\delta})^i \\
&+ \sum_{j=0}^N \sum_{i=1}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} \iota^* (\bar{d}\bar{\delta})^i \\
&+ \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* (\bar{d}\bar{\delta})^i \\
&+ \sum_{j=0}^N \sum_{i=1}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* (\bar{d}\bar{\delta})^i \\
&+ \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} q_j^{(N-1)}(\lambda-1) (d\delta)^{N-i} \iota^* (\bar{d}\bar{\delta})^i \\
&+ \sum_{j=1}^N \sum_{i=1}^{N-j} (-1)^{N-j-i-1} \binom{N-j}{i} q_{j-1}^{(N-1)}(\lambda-1) (d\delta)^{N-i} \iota^* (\bar{d}\bar{\delta})^i \\
&+ \sum_{j=1}^N (-1)^{N-j-1} q_{j-1}^{(N-1)}(\lambda-1) (d\delta)^N \iota^* \\
&- p_0^{(N)}(\lambda; p) \iota^* ((\bar{d}\bar{\delta})^N + (\bar{\delta}\bar{d})^N).
\end{aligned}$$

By $d\iota^* \bar{d} = 0$, the third sum equals

$$\sum_{j=0}^N (-1)^{N-j} p_j^{(N)}(\lambda; p) (\delta d)^N + p_0^{(N)}(\lambda; p) (\bar{d}\bar{\delta})^N.$$

Further simplifications and interchanges of summations yields (5.11).

Now the identities

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) = (\lambda + p - 2N) \alpha_i^{(N)}(\lambda), \quad (5.12)$$

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} S_j(\lambda; N, p) = (\lambda + p) \alpha_i^{(N)}(\lambda), \quad (5.13)$$

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} T_j(\lambda; N, p) = (\lambda + p - 2i) \alpha_i^{(N)}(\lambda) \quad (5.14)$$

for $i = 0, \dots, N$ imply that formula (5.11) is equivalent to

$$D_{2N}^{(p \rightarrow p)}(\lambda) = (\lambda + p - 2N) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (\delta d)^{N-i} \iota^* (\bar{d}\bar{\delta})^i$$

$$\begin{aligned}
& + (\lambda + p) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\
& + \sum_{i=1}^{N-1} (\lambda + p - 2i) \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i.
\end{aligned}$$

This proves the theorem.

It remains to prove the identities (5.12)–(5.14). (5.12) is a direct consequence of the first identity in Lemma 5.1.2 using

$$p_j^{(N)}(\lambda; p) = (\lambda + p - 2N) a_j^{(N)}(\lambda).$$

Next, we observe that

$$\begin{aligned}
r_{j-1}^{(N-1)}(\lambda - 1) &= 2N a_{j-1}^{(N-1)}(\lambda - 1) \quad (\text{by definition of } r_j^{(N-1)}(\lambda)) \\
&= 2j a_j^{(N)}(\lambda) \quad (\text{from the definition of even Gegenbauer coefficients})
\end{aligned}$$

and

$$q_j^{(N-1)}(\lambda - 1) = (2N - 2j) a_j^{(N)}(\lambda) \quad (\text{from the definition of even Gegenbauer coefficients}).$$

Hence

$$S_j(\lambda; N, p) = (\lambda + p) a_j^{(N)}(\lambda).$$

Thus the first identity in Lemma 5.1.2 implies (5.14). Finally, a calculation shows that

$$\begin{aligned}
& q_j^{(N-1)}(\lambda - 1) + q_{j-1}^{(N-1)}(\lambda - 1) \\
&= \frac{N!}{j!(2N+1-2j)!} (-2)^{N+1-j} [j(2\lambda+n) - N(2N+1)] \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n + 1).
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{j=0}^{N-i} (-1)^{N-j} \binom{N-j}{i} \left[q_j^{(N-1)}(\lambda - 1) + q_{j-1}^{(N-1)}(\lambda - 1) \right] \\
&= -(2\lambda + n) \sum_{j=0}^{N-i-1} 2^{N-j} \binom{N-j-1}{i} \frac{N!}{j!(2N-1-2j)!} \prod_{k=j+1}^{N-1} (2\lambda - 4N + 2k + n + 1) \\
&+ N(2N+1) \sum_{j=0}^{N-i} 2^{N+1-j} \binom{N-j}{i} \frac{N!}{j!(2N+1-2j)!} \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n + 1) \\
&= -(2\lambda + n) 2^{2N-1} \binom{N-1}{i} \frac{N!}{(2N-1)!} \\
&\quad \times (\lambda + \frac{n}{2} - 2N + \frac{3}{2})_{N-1} {}_2F_1(-N + \frac{1}{2}, -N + i + 1; \lambda + \frac{n}{2} - 2N + \frac{3}{2}; 1) \\
&+ N(2N+1) 2^{2N+1} \binom{N}{i} \frac{N!}{(2N+1)!} \\
&\quad \times (\lambda + \frac{n}{2} - 2N + \frac{1}{2})_N {}_2F_1(-N - \frac{1}{2}, -N + i; \lambda + \frac{n}{2} - 2N + \frac{1}{2}; 1).
\end{aligned}$$

By the Zhu-Vandermonde formula (5.7), the latter sum equals

$$\begin{aligned} & - (2\lambda + n) 2^{2N-1} \binom{N-1}{i} \frac{N!}{(2N-1)!} \frac{(\lambda + \frac{n}{2} - 2N + \frac{3}{2})_{N-1}}{(\lambda + \frac{n}{2} - 2N + \frac{3}{2})_{N-i-1}} (\lambda + \frac{n}{2} - N + 1)_{N-i-1} \\ & + 2^{2N} \binom{N}{i} \frac{N!}{(2N-1)!} \frac{(\lambda + \frac{n}{2} - 2N + \frac{1}{2})_N}{(\lambda + \frac{n}{2} - 2N + \frac{1}{2})_{N-i}} (\lambda + \frac{n}{2} - N + 1)_{N-i}. \end{aligned}$$

Now simplification gives

$$\begin{aligned} & \frac{[-(\lambda + \frac{n}{2})(N-i) + N(\lambda + \frac{n}{2} - i)]}{(\lambda + \frac{n}{2} - N)} 2^{2N} \frac{(N-1)!}{(2N-1)!} \binom{N}{i} (\lambda + \frac{n}{2} - N)_{N-i} (\lambda + \frac{n}{2} - i - N + \frac{1}{2})_i \\ & = (-1)^i 2i \alpha_i^{(N)}(\lambda). \end{aligned}$$

Thus, we have proved that

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} \left[q_j^{(N-1)}(\lambda-1) + q_{j-1}^{(N-1)}(\lambda-1) \right] = 2i \alpha_i^{(N)}(\lambda).$$

Subtracting this identity from (5.13) proves (5.14). The proof is complete. \square

Now Hodge conjugation relates the even-order families $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^{n-1})$ of the second type to the even-order families $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1})$ of the first type. More precisely, Theorem 4.3.3 implies the following result.

Theorem 5.2.2. *For $p = 1, \dots, n$ and $N \in \mathbb{N}$, the even-order families $D_{2N}^{(p \rightarrow p-1)}(\lambda)$ of the second type can be written in the form*

$$\begin{aligned} D_{2N}^{(p \rightarrow p-1)}(\lambda) &= -(\lambda + n - p - 2N) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\ &\quad - \sum_{i=1}^{N-1} (\lambda + n - p - 2i) \alpha_i^{(N)}(\lambda) (\delta d)^{N-i} \iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\ &\quad - (\lambda + n - p) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (\delta d)^{N-i} \iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \end{aligned} \tag{5.15}$$

with the coefficients $\alpha_i^{(N)}(\lambda)$ defined by (5.5).

Proof. Theorem 5.2.1 and Theorem 4.3.3 imply

$$\begin{aligned} (-1)^{np} D_{2N}^{(p \rightarrow p-1)}(\lambda) &= \star D_{2N}^{(n-p \rightarrow n-p)}(\lambda) \bar{\star} \\ &= (\lambda + n - p) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (\delta d)^{N-i} \star \iota^* \bar{\star} (\bar{\delta}\bar{d})^i \\ &\quad + \sum_{i=1}^{N-1} (\lambda + n - p - 2i) \alpha_i^{(N)}(\lambda) (\delta d)^{N-i} \star \iota^* \bar{\star} (\bar{d}\bar{\delta})^i \\ &\quad + (\lambda + n - p - 2N) \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \star \iota^* \bar{\star} (\bar{d}\bar{\delta})^i \end{aligned}$$

using Lemma 4.3.1/(4). But

$$\star \iota^* \bar{\star} = \iota^* i_{\partial_n} (-1)^{pn+1}$$

on $\Omega^p(\mathbb{R}^n)$ by Lemma 4.3.2. The assertion follows by combining both results. \square

5.3. Odd-order families of the first and second type. We continue with the discussion of odd-order families. We start with odd-order families of the first type. The following result basically restates Theorem 2 in Section 1.

Theorem 5.3.1. *Assume that $N \in \mathbb{N}_0$ and $p = 0, \dots, n-1$. The odd-order family $D_{2N+1}^{(p \rightarrow p)}(\lambda)$ of the first type can be written in the form*

$$\begin{aligned} D_{2N+1}^{(p \rightarrow p)}(\lambda) &= \sum_{i=1}^N \gamma_i^{(N)}(\lambda; p) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta} \bar{d})^i \\ &\quad + (\lambda + p) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d} \bar{\delta})^i \\ &\quad + (\lambda + p - 2N - 1) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d} (\bar{\delta} \bar{d})^i \end{aligned} \quad (5.16)$$

with the coefficients

$$\begin{aligned} \gamma_i^{(N)}(\lambda; p) &= (-1)^i 2^N \frac{N!}{(N+1)(2N+1)!} \binom{N+1}{i} \\ &\quad \times [(\lambda + p - 2N - 1)(N+1)(2\lambda + n - 2i) + (\lambda + n - p)(2N+1)(N-i+1)] \\ &\quad \times \prod_{k=i+1}^N (2\lambda + n - 2k) \prod_{k=1}^{i-1} (2\lambda + n - 2k - 2N - 1), \quad i = 1, \dots, N, \end{aligned} \quad (5.17)$$

and $\beta_i^{(N)}(\lambda)$ as in (5.6).

The coefficients $\gamma_i^{(N)}(\lambda; p)$ can be written in the form

$$\gamma_i^{(N)}(\lambda; p) = \gamma_i^{(N),+}(\lambda; p) + \gamma_i^{(N),-}(\lambda; p) \quad (5.18)$$

with

$$\begin{aligned} \gamma_i^{(N),+}(\lambda; p) &= (-1)^i 2^N \frac{N!}{(2N+1)!} \binom{N+1}{i} \\ &\quad \times (\lambda + p - 2N - 1) \prod_{k=i}^N (2\lambda + n - 2k) \prod_{k=1}^{i-1} (2\lambda + n - 2k - 2N - 1) \end{aligned} \quad (5.19)$$

and

$$\begin{aligned} \gamma_i^{(N),-}(\lambda; p) &= (-1)^i 2^N \frac{N!}{(2N)!} \binom{N}{i} \\ &\quad \times (\lambda + n - p) \prod_{k=i+1}^N (2\lambda + n - 2k) \prod_{k=1}^{i-1} (2\lambda + n - 2k - 2N - 1). \end{aligned} \quad (5.20)$$

The decomposition (5.18) will be important in the proof below.

Remark 5.3.2. *Alternatively, the coefficients $\gamma_i^{(N)}(\lambda; p)$, $i = 1, \dots, N$, can be written in the form*

$$\gamma_i^{(N)}(\lambda; p) = (\lambda + p - 2i)\beta_i^{(N)}(\lambda) - (\lambda + p - 2i + 1)\beta_{i-1}^{(N)}(\lambda). \quad (5.21)$$

Proof. The assertion follows from the identity

$$\begin{aligned} & (\lambda + p - 2N - 1)(N + 1)(2\lambda + n - 2i) + (\lambda + n - p)(2N + 1)(N - i + 1) \\ &= (\lambda + p - 2i + 1)i(2\lambda + n - 2i) + (\lambda + p - 2i)(N - i + 1)(2\lambda + n - 2i - 2N - 1). \end{aligned}$$

We omit the details. \square

Proof. Throughout the following proof we shall use the conventions $q_{-1}^{(N)}(\lambda) = 0$ and $r_{-1}^{(N-1)}(\lambda) = 0$. We start by proving the formula

$$\begin{aligned} D_{2N+1}^{(p \rightarrow p)}(\lambda) &= \sum_{i=0}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} S_j(\lambda; N, p) (\delta d)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\ &\quad + \sum_{i=0}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^i \\ &\quad + \sum_{i=1}^N \sum_{j=-1}^{N-i} (-1)^{N-j-i-1} \binom{N-j}{i} T_j(\lambda; N, p) (\delta d)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \end{aligned} \quad (5.22)$$

with

$$S_j(\lambda; N, p) \stackrel{\text{def}}{=} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda - 1) + q_j^{(N)}(\lambda - 1) \right], \quad j = 0, \dots, N$$

and

$$T_j(\lambda; N, p) \stackrel{\text{def}}{=} \left[p_{j+1}^{(N)}(\lambda; p) + r_j^{(N-1)}(\lambda - 1) - q_j^{(N)}(\lambda - 1) \right], \quad j = -1, \dots, N - 1.$$

First, we use the identities (5.4) to rewrite the family $D_{2N+1}^{(p \rightarrow p)}(\lambda)$ in Theorem 4.1.2 in the form

$$\begin{aligned} D_{2N+1}^{(p \rightarrow p)}(\lambda) &= \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda - 1) \right] (\delta d)^{N-i} d\iota^* i_{\partial_n} \bar{\Delta}^i \\ &\quad + \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda - 1) \right] (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^i \\ &\quad + \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^i \\ &\quad + \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) \Delta^{N-i} (d\iota^* i_{\partial_n} + \iota^* i_{\partial_n} \bar{d}) \bar{\Delta}^i \\ &\quad + \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} q_j^{(N)}(\lambda - 1) (\delta d)^{N-i} d\iota^* i_{\partial_n} \bar{\Delta}^i. \end{aligned} \quad (5.23)$$

Now we simplify this formula. The first sum in (5.23) equals

$$\begin{aligned} & \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\ & + \sum_{j=1}^{N-1} \sum_{i=1}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i. \end{aligned} \quad (5.24)$$

Next, the fourth sum in (5.23) coincides with the sum

$$\begin{aligned} & \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\ & + \sum_{i=1}^N (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\ & + \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) (d\delta)^{N-i} \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^i \\ & + \sum_{i=0}^{N-1} (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^i. \end{aligned} \quad (5.25)$$

By (5.24) and (5.25), the first two sums and the fourth sum in (5.23) combine to

$$\begin{aligned} & \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\ & + \sum_{i=0}^{N-1} (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^i \\ & + \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\ & + \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^i; \end{aligned}$$

we stress that j runs from $j = 0$. We combine the last two sums in the last display by moving in the last sum one factor δ to the right of $\iota^* i_{\partial_n}$ using the second rule in Lemma 4.0.1/(1). The calculation yields

$$\begin{aligned} & \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} (-1)^{N-j-i} \binom{N-j+1}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\ & - \sum_{j=0}^{N-1} \left[p_{j+1}^{(N)}(\lambda) + r_j^{(N-1)}(\lambda-1) \right] (d\delta)^j d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^{N-j} \\ & + p_0^{(N)}(\lambda; p) \iota^* i_{\partial_n} \bar{d}(\bar{\delta}\bar{d})^N. \end{aligned}$$

Summarizing the above results, we find that the first four sums in (5.23) combine to

$$\begin{aligned}
& \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\
& + \sum_{i=0}^{N-1} (-1)^{N-i} \binom{N}{i} p_0^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d} (\bar{\delta}\bar{d})^i \\
& + \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} (-1)^{N-j-i} \binom{N-j+1}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\
& - \sum_{j=0}^{N-1} \left[p_{j+1}^{(N)}(\lambda) + r_j^{(N-1)}(\lambda-1) \right] (d\delta)^j d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^{N-j} \\
& + p_0^{(N)}(\lambda; p) \iota^* i_{\partial_n} \bar{d} (\bar{\delta}\bar{d})^N \\
& + \sum_{j=1}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d} (\bar{\delta}\bar{d})^i.
\end{aligned}$$

Now we perform an index shift in the third sum and summarize the second sum, the fifth term and the sixth sum. We obtain

$$\begin{aligned}
& \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\
& + \sum_{j=-1}^{N-2} \sum_{i=1}^{N-j-1} (-1)^{N-j-i-1} \binom{N-j}{i} \left[p_{j+1}^{(N)}(\lambda) + r_j^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\
& + \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d} (\bar{\delta}\bar{d})^i \\
& - \sum_{j=0}^{N-1} \left[p_{j+1}^{(N)}(\lambda) + r_j^{(N-1)}(\lambda-1) \right] (d\delta)^j d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^{N-j}.
\end{aligned}$$

The last sum can be regarded as the contribution $i = N - j$ in the second sum. Hence these terms combine to

$$\begin{aligned}
& \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} (-1)^{N-j-i-1} \binom{N-j}{i} \left[p_{j+1}^{(N)}(\lambda) + r_j^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\
& + \sum_{i=1}^N (-1)^{N-i} \binom{N+1}{i} p_0^{(N)}(\lambda; p) (\delta d)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i
\end{aligned}$$

After interchanges of the summations we find

$$\sum_{i=0}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i$$

$$\begin{aligned}
& + \sum_{i=1}^N \sum_{j=0}^{N-i} (-1)^{N-j-i-1} \binom{N-j}{i} \left[p_{j+1}^{(N)}(\lambda) + r_j^{(N-1)}(\lambda-1) \right] (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\
& + \sum_{i=1}^N (-1)^{N-i} \binom{N+1}{i} p_0^{(N)}(\lambda; p) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i \\
& + \sum_{i=0}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) (\delta d)^{N-i} \iota^* i_{\partial_n} \bar{d} (\bar{\delta}\bar{d})^i.
\end{aligned} \tag{5.26}$$

Finally, the fifth sum in (5.23) expands as

$$\begin{aligned}
& \sum_{i=0}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} q_j^{(N)}(\lambda-1) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{d}\bar{\delta})^i \\
& + \sum_{i=1}^N \sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} q_j^{(N)}(\lambda-1) (d\delta)^{N-i} d\iota^* i_{\partial_n} (\bar{\delta}\bar{d})^i.
\end{aligned} \tag{5.27}$$

Combining (5.26) and (5.27), we obtain the formula (5.22).

In order to complete the proof of the theorem, it remains to verify the identities

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} \left[p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) + q_j^{(N)}(\lambda-1) \right] = (\lambda+p)\beta_i^{(N)}(\lambda) \tag{5.28}$$

and

$$\sum_{j=0}^{N-i} (-1)^{N-j-i} \binom{N-j}{i} p_j^{(N)}(\lambda; p) = (\lambda+p-2N-1)\beta_i^{(N)}(\lambda) \tag{5.29}$$

for $i = 0, \dots, N$, and

$$\sum_{j=-1}^{N-i} (-1)^{N-j-i-1} \binom{N-j}{i} \left[p_{j+1}^{(N)}(\lambda; p) + r_j^{(N-1)}(\lambda-1) - q_j^{(N)}(\lambda-1) \right] = \gamma_i^{(N)}(\lambda; p) \tag{5.30}$$

for $i = 1, \dots, N$. But since by definition

$$p_j^{(N)}(\lambda; p) = (\lambda+p-2N-1)b_j^{(N)}(\lambda),$$

the identity (5.29) is a direct consequence of the second part of Lemma 5.1.2. Next, we observe that

$$\begin{aligned}
r_{j-1}^{(N-1)}(\lambda-1) &= 2Nb_{j-1}^{(N-1)}(\lambda-1) \quad (\text{by definition of } r_j^{(N-1)}(\lambda)) \\
&= 2jb_j^{(N)}(\lambda) \quad (\text{from the definition of odd Gegenbauer coefficients})
\end{aligned}$$

and

$$q_j^{(N)}(\lambda-1) = (2N-2j+1)b_j^{(N)}(\lambda) \quad (\text{from the definition of odd Gegenbauer coefficients}).$$

Hence

$$p_j^{(N)}(\lambda; p) + r_{j-1}^{(N-1)}(\lambda-1) + q_j^{(N)}(\lambda-1) = (\lambda+p)b_j^{(N)}(\lambda)$$

and (5.28) follows from the second identity in Lemma 5.1.2. In order to prove (5.30), we first calculate

$$p_{j+1}^{(N)}(\lambda; p) + r_j^{(N-1)}(\lambda-1) - q_j^{(N)}(\lambda-1)$$

$$\begin{aligned}
&= (\lambda + p - 2N + 1 + 2j)b_{j+1}^{(N)}(\lambda) - (2N - 2j + 1)b_j^{(N)}(\lambda) \\
&= -[(\lambda + p - 2N - 1)(N + 1) + (\lambda + n - p)(j + 1)] \\
&\quad \times (-2)^{N-j} \frac{N!}{(j+1)!(2N-2j)!} \prod_{k=j+1}^{N-1} (2\lambda + n - 4N + 2k - 1) \quad (5.31)
\end{aligned}$$

for $j = 0, \dots, N-1$. This formula extends to $j = -1$. Now we split the left-hand side of (5.30) into two parts:

$$\begin{aligned}
&\sum_{j=-1}^{N-i} (-1)^{N-i-j} (-2)^{N-j} \binom{N-j}{i} \frac{(N+1)!}{(j+1)!(2N-2j)!} \\
&\quad \times (\lambda + p - 2N - 1) \prod_{k=j+1}^{N-1} (2\lambda + n - 4N + 2k - 1)
\end{aligned}$$

and

$$\sum_{j=0}^{N-i} (-1)^{N-i-j} (-2)^{N-j} \binom{N-j}{i} \frac{N!}{j!(2N-2j)!} (\lambda + n - p) \prod_{k=j+1}^{N-1} (2\lambda + n - 4N + 2k - 1);$$

note that the second sum runs from $j = 0$. For $1 \leq i \leq N$ both sums are hypergeometric and we find the respective formulas

$$\begin{aligned}
&(-1)^i 4^N \binom{N+1}{i} \frac{N!}{(2N+1)!} (\lambda + p - 2N - 1) \left(\lambda + \frac{n}{2} - 2N - \frac{1}{2}\right)_N \\
&\quad \times {}_2F_1\left(-N - \frac{1}{2}, i - N - 1; \lambda + \frac{n}{2} - 2N - \frac{1}{2}; 1\right)
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^i 2^{2N-1} \binom{N}{i} \frac{N!}{(2N)!} (\lambda + n - p) \left(\lambda + \frac{n}{2} - 2N + \frac{1}{2}\right)_{N-1} \\
&\quad \times {}_2F_1\left(-N + \frac{1}{2}, i - N; \lambda + \frac{n}{2} - 2N + \frac{1}{2}; 1\right).
\end{aligned}$$

Application of the Zhu-Vandermonde formula (5.7) yields

$$\begin{aligned}
&(-1)^i 4^N \binom{N+1}{i} \frac{N!}{(2N+1)!} (\lambda + p - 2N - 1) \left(\lambda + \frac{n}{2} - N\right)_{N+1-i} \frac{\left(\lambda + \frac{n}{2} - 2N - \frac{1}{2}\right)_N}{\left(\lambda + \frac{n}{2} - 2N - \frac{1}{2}\right)_{N+1-i}} \\
&= \gamma_i^{(N),+}(\lambda; p)
\end{aligned}$$

and

$$(-1)^i 2^{2N-1} \binom{N}{i} \frac{N!}{(2N)!} (\lambda + n - p) \left(\lambda + \frac{n}{2} - N\right)_{N-i} \frac{\left(\lambda + \frac{n}{2} - 2N + \frac{1}{2}\right)_{N-1}}{\left(\lambda + \frac{n}{2} - 2N + \frac{1}{2}\right)_{N-i}} = \gamma_i^{(N),-}(\lambda; p)$$

(see (5.19), (5.20)). This proves (5.30) for $1 \leq i \leq N$. \square

Finally, Hodge conjugation relates the odd-order families $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p-1}(\mathbb{R}^{n-1})$ of the second type to the odd-order families $\Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1})$ of the first type. More precisely, Theorem 4.3.3 implies the following result.

Theorem 5.3.3. *Assume that $N \in \mathbb{N}_0$ and $p = 1, \dots, n$. The odd-order families $D_{2N+1}^{(p \rightarrow p-1)}(\lambda)$ of the second type can be written in the form*

$$\begin{aligned} D_{2N+1}^{(p \rightarrow p-1)}(\lambda) = & - \sum_{i=1}^N \gamma_i^{(N)}(\lambda; n-p) (\delta d)^{N-i} \delta \iota^* (\bar{d} \bar{\delta})^i \\ & + (\lambda + n - p - 2N - 1) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^* \bar{\delta} (\bar{d} \bar{\delta})^i \\ & - (\lambda + n - p) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (\delta d)^{N-i} \delta \iota^* (\bar{\delta} \bar{d})^i \end{aligned} \quad (5.32)$$

with the coefficients $\beta_i^{(N)}(\lambda)$ and $\gamma_i^{(N)}(\lambda; p)$ as defined in (5.6) and (5.17), respectively.

Proof. Theorem 5.3.1 and Theorem 4.3.3 imply

$$\begin{aligned} (-1)^{np} D_{2N+1}^{(p \rightarrow p-1)}(\lambda) &= \star D_{2N+1}^{(n-p \rightarrow n-p)}(\lambda) \bar{\star} \\ &= \sum_{i=1}^N \gamma_i^{(N)}(\lambda; n-p) (\delta d)^{N-i} \star d \iota^* i_{\partial_n} \bar{\star} (\bar{d} \bar{\delta})^i \\ &\quad + (\lambda + n - p - 2N - 1) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (d\delta)^{N-i} \star \iota^* i_{\partial_n} \bar{d} \bar{\star} (\bar{d} \bar{\delta})^i \\ &\quad + (\lambda + n - p) \sum_{i=0}^N \beta_i^{(N)}(\lambda) (\delta d)^{N-i} \star d \iota^* i_{\partial_n} \bar{\star} (\bar{\delta} \bar{d})^i \end{aligned}$$

using Lemma 4.3.1/(4). But Lemma 4.3.1/(2) and Lemma 4.3.2 give

$$\star d \iota^* i_{\partial_n} \bar{\star} = \delta \star \iota^* i_{\partial_n} \bar{\star} (-1)^{n-p} = \delta \iota^* (-1)^{np+1}$$

and

$$\star \iota^* i_{\partial_n} \bar{d} \bar{\star} = \star \iota^* i_{\partial_n} \bar{\star} \bar{\delta} (-1)^p = \iota^* \bar{\delta} (-1)^{pn}$$

on $\Omega^p(\mathbb{R}^n)$. Combining these results proves the assertion. \square

5.4. Operators of the third and fourth type. In the present section, we derive geometrical formulas for the conformal symmetry breaking operators of the third and the fourth type.

Theorem 5.4.1. *The even-order operators $D_{2N}^{(0 \rightarrow 1)}$, $N \in \mathbb{N}$, of the third type can be written in the form*

$$D_{2N}^{(0 \rightarrow 1)} = \sum_{i=0}^{N-1} \beta_j^{(N-1)}(2N-1) (d\delta)^{N-i-1} d \iota^* i_{\partial_n} \bar{d} (\bar{\delta} \bar{d})^i$$

with the coefficients $\beta_j^{(N)}(\lambda)$ defined by (5.6).

Proof. By the definition of $D_{2N}^{(0 \rightarrow 1)}$ (Theorem 4.4.1) and the expansion (5.4) of $\iota^* \partial_n^{2N-2j-1}$, we obtain

$$D_{2N}^{(0 \rightarrow 1)} = \sum_{j=0}^{N-1} \sum_{i=0}^{N-j-1} (-1)^{N-j-i-1} \binom{N-j-1}{i} b_j^{(N-1)}(2N-1) d (\delta d)^{N-i-1} \iota^* i_{\partial_n} \bar{d} (\bar{\delta} \bar{d})^i.$$

Interchanging summations and applying Lemma 5.1.2 completes the proof. \square

Theorem 5.4.2. *The odd-order operators $D_{2N+1}^{(0 \rightarrow 1)}$, $N \in \mathbb{N}_0$, of the third type can be written in the form*

$$D_{2N+1}^{(0 \rightarrow 1)} = \sum_{i=0}^N \alpha_j^{(N)}(2N)(d\delta)^{N-i} \iota^* (\bar{\delta} \bar{d})^i$$

with the coefficients $\alpha_j^{(N)}(\lambda)$ defined by (5.5).

Proof. By the definition of $D_{2N+1}^{(0 \rightarrow 1)}$ (Theorem 4.4.2) and the expansion (5.3) of $\iota^* \partial_n^{2N-2j}$, we obtain

$$D_{2N+1}^{(0 \rightarrow 1)} = \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i} \binom{N-j}{i} a_j^{(N)}(2N) d(\delta d)^{N-i} \iota^* (\bar{\delta} \bar{d})^i.$$

Interchanging summations and applying Lemma 5.1.2 completes the proof. \square

Remark 5.4.3. *In view of $D_N^{(0 \rightarrow 1)} = d\dot{D}_{N-1}^{(0 \rightarrow 0)}(N-1)$, the same results also follow from the geometrical formula for $D_N^{(0 \rightarrow 0)}(\lambda)$.*

We continue with the derivation of the analogous formulas for the conformal symmetry breaking operators of the fourth type.

Theorem 5.4.4. *The even-order operators $D_{2N}^{(n \rightarrow n-2)}$, $N \in \mathbb{N}$, of the fourth type can be written in the form*

$$D_{2N}^{(n \rightarrow n-2)} = \sum_{i=0}^{N-1} \beta_j^{(N-1)}(2N-1)(\delta d)^{N-i-1} \delta \iota^* \bar{\delta} (\bar{d} \bar{\delta})^i,$$

with the coefficients $\beta_j^{(N)}(\lambda)$ defined by (5.6).

Proof. By the definition of $D_{2N}^{(n \rightarrow n-2)}$ (Theorem 4.5.1) and the expansion (5.4) of $\iota^* i_{\partial_n} \partial_n^{2N-2j-1}$, we obtain

$$D_{2N}^{(n \rightarrow n-2)} = \sum_{j=0}^{N-1} \sum_{i=0}^{N-j-1} (-1)^{N-j-i-1} \binom{N-j-1}{i} b_j^{(N-1)}(2N-1) \delta (d\delta)^{N-i-1} \iota^* \bar{\delta} (\bar{d} \bar{\delta})^i.$$

Interchanging summations and applying Lemma 5.1.2 completes the proof. \square

Theorem 5.4.5. *The odd-order operators $D_{2N+1}^{(n \rightarrow n-2)}$, $N \in \mathbb{N}_0$, of the fourth type can be written in the form*

$$D_{2N+1}^{(n \rightarrow n-2)} = - \sum_{i=0}^N \alpha_i^{(N)}(2N)(\delta d)^{N-i} \delta \iota^* i_{\partial_n} (\bar{d} \bar{\delta})^i,$$

with the coefficients $\beta_j^{(N)}(\lambda)$ defined by (5.6).

Proof. By the definition of $D_{2N+1}^{(n \rightarrow n-2)}$ (Theorem 4.5.2) and the expansion (5.3) of $\iota^* i_{\partial_n} \partial_n^{2N-2j}$, we obtain

$$D_{2N+1}^{(n \rightarrow n-2)} = \sum_{j=0}^N \sum_{i=0}^{N-j} (-1)^{N-j-i+1} \binom{N-j}{i} a_j^{(N)}(2N) \delta (d\delta)^{N-i} \iota^* i_{\partial_n} (\bar{d} \bar{\delta})^i.$$

Interchanging summations and applying Lemma 5.1.2 completes the proof. \square

Remark 5.4.6. *In view of $D_N^{(n \rightarrow n-2)} = \delta \dot{D}_{N-1}^{(n \rightarrow n-1)}(N-1)$, the same results also follow from the geometrical formula for $D_N^{(n \rightarrow n-1)}(\lambda)$.*

6. FACTORIZATION IDENTITIES FOR CONFORMAL SYMMETRY BREAKING OPERATORS

In the present section, we discuss natural identities which describe factorizations of conformal symmetry breaking operators into products of conformally covariant operators. There are two types of such factorizations. The main factorizations contain Branson-Gover operators as factors and the supplementary factorizations contain exterior differentials and co-differentials as factors. The main factorizations generalize corresponding results in [J09].

As consequences, we shall see that the conformal symmetry breaking operators serve as a natural organizing package for several important differential operators acting on differential forms. In particular, we shall prove that for the Euclidean metric g_0 they naturally capture the Branson-Gover, gauge companion and Q -curvature operators [BG05].

6.1. Branson-Gover, gauge companion and Q -curvature operators. We first recall some basic facts on these constructions. For more details we refer to [BG05], [G04] and [AG11]. The Branson-Gover operators $L_{2N}^{(p)}(g)$ on a Riemannian manifold (M^n, g) of dimension $n \geq 3$ are conformally covariant differential operators on $\Omega^p(M)$ of order $2N \geq 2$. They satisfy the intertwining relation

$$e^{(\frac{n}{2}-p+N)\sigma} \hat{L}_{2N}^{(p)}(\omega) = L_{2N}^{(p)}(e^{(\frac{n}{2}-p-N)\sigma}\omega), \quad \omega \in \Omega^p(M), \quad (6.1)$$

where $L_{2N}^{(p)} = L_{2N}^{(p)}(g)$ and $\hat{L}_{2N}^{(p)} = L_{2N}^{(p)}(\hat{g})$ are the respective Branson-Gover operators for the metrics g and $\hat{g} = e^{2\sigma}g$.¹² For even n and $p \leq \frac{n}{2} - 1$, the operators $L_{n-2p}^{(p)}$ will be called the *critical* Branson-Gover operators. The operator $L_{2N}^{(0)}$ on $C^\infty(M)$ reduces to a constant multiple of the GJMS-operator of order $2N$. In particular, in even dimension n , the critical Branson-Gover operator $L_n^{(0)}$ is a constant multiple of the critical GJMS-operator P_n .

For general metrics, even n and $p \leq \frac{n}{2} - 1$, we also consider the gauge companion operator

$$G_{n-2p+1}^{(p)} : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$$

and the (critical) Q -curvature operator

$$Q_{n-2p}^{(p)} : \Omega^p(M)|_{\ker(d)} \rightarrow \Omega^p(M).$$

These operators appear in the factorization identities

$$L_{n-2p}^{(p)} = (n-2p)G_{n-2p-1}^{(p+1)}d \quad \text{and} \quad G_{n-2p-1}^{(p+1)} = \delta Q_{n-2p-2}^{(p+1)}.$$

The operators $Q_{n-2p}^{(p)}$ generalize Branson's Q -curvature Q_n on functions. The property

$$L_{n-2p}^{(p)} \sim \delta Q_{n-2p-2}^{(p+1)}d \quad (6.2)$$

is known as the double factorization property of the critical Branson-Gover operators.

¹²We suppress the conditions on the dimension n and the order $2N$ which guarantee the existence of the operators.

Under conformal changes $g \mapsto e^{2\sigma}g$, these operators transform according to

$$e^{(n-2p)\sigma} \widehat{G}_{n-2p-1}^{(p+1)}(\omega) = G_{n-2p-1}^{(p+1)}(\omega) + i_{\text{grad}(\sigma)} L_{n-2p-2}^{(p+1)}(\omega) \quad (6.3)$$

and

$$e^{(n-2p)\sigma} \widehat{Q}_{n-2p}^{(p)}(\omega) = Q_{n-2p}^{(p)}(\omega) + L_{n-2p}^{(p)}(\sigma\omega). \quad (6.4)$$

Since the critical Branson-Gover operator $L_{n-2p}^{(p)}$ annihilates closed forms, the transformation property (6.3) implies that the restriction of $G_{n-2p-1}^{(p+1)}$ to closed forms is conformally covariant.

On the Euclidean space (\mathbb{R}^{n-1}, g_0) , these operators take the following form.¹³ First, the Branson-Gover operators are given by the explicit formula

$$L_{2N}^{(p)} = \left(\frac{n-1}{2} - p + N\right)(d\delta)^N + \left(\frac{n-1}{2} - p - N\right)(d\delta)^N. \quad (6.5)$$

Now assume that $n-1$ is even and $n-1-2p \geq 2$. Then the critical Branson-Gover operator

$$L_{n-1-2p}^{(p)} = (n-1-2p)\delta(d\delta)^{\frac{n-3}{2}-p}d$$

factors through

$$G_{n-2-2p}^{(p+1)} \stackrel{\text{def}}{=} \delta(d\delta)^{\frac{n-3}{2}-p} : \Omega^{p+1}(\mathbb{R}^{n-1}) \rightarrow \Omega^p(\mathbb{R}^{n-1})$$

and the critical Q -curvature operator

$$Q_{n-3-2p}^{(p+1)} \stackrel{\text{def}}{=} (d\delta)^{\frac{n-3}{2}-p} : \Omega^{p+1}(\mathbb{R}^{n-1}) \rightarrow \Omega^{p+1}(\mathbb{R}^{n-1}).$$

The conformal covariance (6.1) of the Branson-Gover operators $L_{2N}^{(p)}$ implies the equivariance of $L_{2N}^{(p)}(g_0)$ under the conformal group of \mathbb{R}^n . In fact, assume that γ is a conformal diffeomorphism of the Euclidean metric g_0 on \mathbb{R}^n , i.e., $\gamma_*(g_0) = e^{2\Phi_\gamma}g_0$ for some $\Phi_\gamma \in C^\infty(\mathbb{R}^n)$. Then (6.1) implies

$$e^{(\frac{n}{2}-p+N)\Phi_\gamma} L_{2N}^{(p)}(\gamma_*(g_0)) = L_{2N}^{(p)}(g_0) e^{(\frac{n}{2}-p-N)\Phi_\gamma}.$$

But, by the naturality of the Branson-Gover operators, we have

$$L_{2N}^{(p)}(\gamma_*(g_0)) = \gamma_* L_{2N}^{(p)}(g_0) \gamma^*.$$

Hence

$$e^{(\frac{n}{2}-p+N)\Phi_\gamma} \gamma_* L_{2N}^{(p)}(g_0) = L_{2N}^{(p)}(g_0) e^{(\frac{n}{2}-p-N)\Phi_\gamma} \gamma_*.$$

In other words, the operator $L_{2N}^{(p)}(g_0)$ satisfies the intertwining property

$$\pi_{\frac{n}{2}-p+N}^{(p)}(\gamma) L_{2N}^{(p)}(g_0) = L_{2N}^{(p)}(g_0) \pi_{\frac{n}{2}-p-N}^{(p)}(\gamma), \quad (6.6)$$

where

$$\pi_\lambda^{(p)}(\gamma) = e^{\lambda\Phi_\gamma} \gamma_* : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^n).$$

We also recall that $\pi_{-\lambda-p}^{(p)} = \pi_{\lambda,p}^\vee$ (see (4.4)). This is the non-compact analog of (2.18).

¹³In contrast to the general theory, in the present case the gauge companion operators and the Q -curvature operators are defined on all forms.

6.2. Main factorizations. Here we show that for special values of the parameter λ the families of the first and second type factorize as compositions of respective lower-order families and Branson-Gover operators acting on forms on \mathbb{R}^{n-1} and \mathbb{R}^n . In particular, the Branson-Gover operators on forms on \mathbb{R}^{n-1} and \mathbb{R}^n are naturally captured by both types of even-order conformal symmetry breaking operators.

We start with the discussion of even-order families of the first type.

Theorem 6.2.1. *Let $N \in \mathbb{N}$ and $p = 0, \dots, n-1$. Then the even-order families of the first type satisfy the factorization identities*

$$\left(\frac{n}{2} - p + k\right) D_{2N}^{(p \rightarrow p)}\left(k - \frac{n}{2}\right) = D_{2N-2k}^{(p \rightarrow p)}\left(-k - \frac{n}{2}\right) \circ \bar{L}_{2k}^{(p)} \quad (6.7)$$

and

$$\left(\frac{n-1}{2} - p - k\right) D_{2N}^{(p \rightarrow p)}\left(2N - k - \frac{n-1}{2}\right) = L_{2k}^{(p)} \circ D_{2N-2k}^{(p \rightarrow p)}\left(2N - k - \frac{n-1}{2}\right) \quad (6.8)$$

for $k = 1, \dots, N-1$, $N \geq 2$. Moreover, in the extremal case $k = N$, we have

$$D_{2N}^{(p \rightarrow p)}\left(N - \frac{n-1}{2}\right) = -L_{2N}^{(p)} \iota^* \quad \text{and} \quad D_{2N}^{(p \rightarrow p)}\left(N - \frac{n}{2}\right) = -\iota^* \bar{L}_{2N}^{(p)}. \quad (6.9)$$

By Hodge conjugation, Theorem 6.2.1 implies the following result for families of the second type.

Theorem 6.2.2. *Let $N \in \mathbb{N}$ and $p = 1, \dots, n$. Then the even-order families of the second type satisfy the factorization identities*

$$\left(\frac{n}{2} - p - k\right) D_{2N}^{(p \rightarrow p-1)}\left(k - \frac{n}{2}\right) = D_{2N-2k}^{(p \rightarrow p-1)}\left(-k - \frac{n}{2}\right) \circ \bar{L}_{2k}^{(p)} \quad (6.10)$$

and

$$\left(\frac{n+1}{2} - p + k\right) D_{2N}^{(p \rightarrow p-1)}\left(2N - k - \frac{n-1}{2}\right) = L_{2k}^{(p-1)} \circ D_{2N-2k}^{(p \rightarrow p-1)}\left(2N - k - \frac{n-1}{2}\right) \quad (6.11)$$

for $k = 1, \dots, N-1$, $N \geq 2$. Moreover, in the extremal case $k = N$, we have

$$D_{2N}^{(p \rightarrow p-1)}\left(N - \frac{n-1}{2}\right) = -L_{2N}^{(p-1)} \iota^* i_{\partial_n} \quad \text{and} \quad D_{2N}^{(p \rightarrow p-1)}\left(N - \frac{n}{2}\right) = -\iota^* i_{\partial_n} \bar{L}_{2N}^{(p)}. \quad (6.12)$$

Some comments concerning these results are in order.

The relation (2.18) shows that Theorem 6.2.1 and Theorem 6.2.2 are compatible with the respective equivariance properties of the Branson-Gover operators and the conformal symmetry breaking operators. Indeed, the relation

$$d\pi_{\lambda-2N,p}^{\vee}(X) D_{2N}^{(p \rightarrow p)}(\lambda) = D_{2N}^{(p \rightarrow p)}(\lambda) d\pi_{\lambda,p}^{\vee}(X), \quad X \in \mathfrak{g}'(\mathbb{R})$$

(see Theorem 4.8) implies

$$d\pi_{-\frac{n-1}{2}-N,p}^{\vee}(X) D_{2N}^{(p \rightarrow p)}\left(N - \frac{n-1}{2}\right) = D_{2N}^{(p \rightarrow p)}\left(N - \frac{n-1}{2}\right) d\pi_{-\frac{n-1}{2}+N,p}^{\vee}(X).$$

Hence, by (6.9) and (2.18), we find

$$d\pi_{\frac{n-1}{2}+N-p}^{\vee(p)}(X) L_{2N}^{(p)} = L_{2N}^{(p)} d\pi_{\frac{n-1}{2}-N-p}^{\vee(p)}(X)$$

which fits with (6.6).

The relations in Theorem 6.2.1 generalize results in [J09] ($p = 0$). An important difference to the results in [J09] is that the above identities contain non-trivial numerical factors on the left-hand sides.

Theorem 6.2.1 and Theorem 6.2.2 are also suggested by the multiplicity free character identities (see Proposition 2.3.1) and the equivariance of the Branson-Gover operators.

From this perspective, their validity can be regarded as a cross-check of the explicit formulas for the families displayed in Theorem 5.2.1 and Theorem 5.2.2. The latter argument, however, does not yield the constant factors which relate both sides of the identities. These factors can also be determined by the following formal argument. We note that the right-hand sides of (6.7) and (6.8) are *quadratic* in p (each factor is linear in p). Therefore, the left-hand side must contain an additional linear factor in p which easily can be read off. We stress that these constant factors actually may vanish. For instance, the left-hand side of (6.10) vanishes if n is even and $k = p - \frac{n}{2} \geq 1$. For these parameters, the right-hand side contains the operator $\bar{L}_{2p-n}^{(p)} = (\bar{d}\bar{\delta})^{p-\frac{n}{2}}$.

The following proof derives the assertions from the geometric formulas in Section 5.

Proof of Theorem 6.2.1. We first prove (6.7). By (6.5), we have

$$\bar{L}_{2k}^{(p)} = \left(\frac{n}{2} - p + k\right)(\bar{d}\bar{\delta})^k + \left(\frac{n}{2} - p - k\right)(\bar{d}\bar{\delta})^k. \quad (6.13)$$

Hence Theorem 5.2.1 implies

$$\begin{aligned} D_{2N}^{(p \rightarrow p)}\left(k - \frac{n}{2}\right) &= \left(k - \frac{n}{2} + p\right) \sum_{i=0}^N \alpha_i^{(N)}\left(k - \frac{n}{2}\right) (\delta d)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + \sum_{i=1}^{N-1} \left(k - \frac{n}{2} + p - 2i\right) \alpha_i^{(N)}\left(k - \frac{n}{2}\right) (\delta d)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + \left(k - \frac{n}{2} + p - 2N\right) \sum_{i=0}^N \alpha_i^{(N)}\left(k - \frac{n}{2}\right) (\delta d)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \end{aligned}$$

and

$$\begin{aligned} D_{2N-2k}^{(p \rightarrow p)}\left(-k - \frac{n}{2}\right) \circ \bar{L}_{2k}^{(p)} &= \left(\frac{n}{2} - p - k\right)\left(-k - \frac{n}{2} + p\right) \sum_{i=k}^N \alpha_{i-k}^{(N-k)}\left(-k - \frac{n}{2}\right) (\delta d)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + \left(\frac{n}{2} - p + k\right)\left(-k - \frac{n}{2} + p\right) \alpha_0^{(N-k)}\left(-k - \frac{n}{2}\right) (\delta d)^{N-k} \iota^*(\bar{d}\bar{\delta})^k \\ &\quad + \left(\frac{n}{2} - p + k\right) \sum_{i=k+1}^{N-1} \left(k - \frac{n}{2} + p - 2i\right) \alpha_{i-k}^{(N-k)}\left(-k - \frac{n}{2}\right) (\delta d)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + \left(\frac{n}{2} - p + k\right)\left(k - \frac{n}{2} + p - 2N\right) \sum_{i=k}^N \alpha_{i-k}^{(N-k)}\left(-k - \frac{n}{2}\right) (\delta d)^{N-i} \iota^*(\bar{d}\bar{\delta})^i. \end{aligned}$$

We note that, in the last sum, the term in the second line originates from the composition of the corresponding first sum in (5.10) with $\bar{L}_{2k}^{(p)}$. It can be merged with the third line by extending the sum to run from $i = k$. Thus, it remains to prove

$$\begin{aligned} \alpha_i^{(N)}\left(k - \frac{n}{2}\right) &= 0, \quad i = 0, \dots, k-1, \\ \alpha_i^{(N)}\left(k - \frac{n}{2}\right) &= \alpha_{i-k}^{(N-k)}\left(-k - \frac{n}{2}\right), \quad i = k, \dots, N. \end{aligned}$$

Both equalities are direct consequences of the definition (5.5) of the coefficients $\alpha_i^{(N)}(\lambda)$. We omit the details of the calculation. This proves (6.7).

Next, we prove (6.8). By (6.5), we have

$$L_{2k}^{(p)} = \left(\frac{n-1}{2} - p + k\right)(\delta d)^k + \left(\frac{n-1}{2} - p - k\right)(\delta d)^k. \quad (6.14)$$

Hence Theorem 5.2.1 implies

$$\begin{aligned} D_{2N}^{(p \rightarrow p)}(2N - k - \frac{n-1}{2}) &= (2N - k - \frac{n-1}{2} + p) \sum_{i=0}^N \alpha_i^{(N)}(2N - k - \frac{n-1}{2})(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + \sum_{i=1}^{N-1} (2N - k - \frac{n-1}{2} + p - 2i) \alpha_i^{(N)}(2N - k - \frac{n-1}{2})(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + (-k - \frac{n-1}{2} + p) \sum_{i=0}^N \alpha_i^{(N)}(2N - k - \frac{n-1}{2})(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \end{aligned}$$

and

$$\begin{aligned} L_{2k}^{(p)} \circ D_{2N-2k}^{(p \rightarrow p)}(2N - k - \frac{n-1}{2}) &= (\frac{n-1}{2} - p - k)(2N - k - \frac{n-1}{2} + p) \sum_{i=0}^{N-k} \alpha_i^{(N-k)}(2N - k - \frac{n-1}{2})(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + (\frac{n-1}{2} - p - k) \sum_{i=1}^{N-k-1} (2N - k - \frac{n-1}{2} + p - 2i) \alpha_i^{(N-k)}(2N - k - \frac{n-1}{2})(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + (\frac{n-1}{2} - p + k)(k - \frac{n-1}{2} + p) \sum_{i=0}^{N-k} \alpha_i^{(N-k)}(2N - k - \frac{n-1}{2})(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\ &\quad + (\frac{n-1}{2} - p - k)(k - \frac{n-1}{2} + p) \alpha_{N-k}^{(N-k)}(2N - k - \frac{n-1}{2})(d\delta)^k \iota^*(\bar{d}\bar{\delta})^{N-k}. \end{aligned}$$

We note that, in the last sum, the term in the fourth line originates from the composition of the corresponding third sum in (5.10) with $L_{2k}^{(p)}$. It can be merged with the second line by extending the sum to run up to $i = N - k$. Thus, it remains to prove

$$\begin{aligned} \alpha_i^{(N)}(2N - k - \frac{n-1}{2}) &= 0, \quad i = N - k + 1, \dots, N, \\ \alpha_i^{(N)}(2N - k - \frac{n-1}{2}) &= \alpha_i^{(N-k)}(2N - k - \frac{n-1}{2}), \quad i = 0, \dots, N - k. \end{aligned}$$

But both equalities are direct consequences of the definition of the coefficients $\alpha_i^{(N)}(\lambda)$. We omit the details of the calculations. This proves (6.8).

It only remains to prove (6.9). Theorem 5.2.1 and (6.14) yield

$$D_{2N}^{(p \rightarrow p)}(N - \frac{n-1}{2}) = (N - \frac{n-1}{2} + p)(d\delta)^N \iota^* + (-N - \frac{n-1}{2} + p)(d\delta)^N \iota^* = -L_{2N}^{(p)} \iota^*$$

using

$$\begin{aligned} \alpha_i^{(N)}(N - \frac{n-1}{2}) &= 0 \quad \text{for } i \geq 1, \\ \alpha_0^{(N)}(N - \frac{n-1}{2}) &= 1. \end{aligned}$$

This proves the first identity. Similarly, Theorem 5.2.1 and (6.13) imply

$$D_{2N}^{(p \rightarrow p)}(N - \frac{n}{2}) = (N - \frac{n}{2} + p) \iota^*(\bar{d}\bar{\delta})^N + (-N - \frac{n}{2} + p) \iota^*(\bar{d}\bar{\delta})^N = -\iota^* \bar{L}_{2N}^{(p)}$$

using

$$\begin{aligned} \alpha_i^{(N)}(N - \frac{n}{2}) &= 0 \quad \text{for } i \leq N - 1, \\ \alpha_N^{(N)}(N - \frac{n}{2}) &= 1. \end{aligned}$$

This proves the second identity. \square

We continue with the **proof of Theorem 6.2.2.**

The relation (6.7) implies

$$\left(-\frac{n}{2} + p + k\right) \star D_{2N}^{(n-p \rightarrow n-p)}\left(k - \frac{n}{2}\right) \bar{\star} = \left(\star D_{2N-2k}^{(n-p \rightarrow n-p)}\left(-k - \frac{n}{2}\right) \bar{\star}\right) \bar{\star}^2 \left(\bar{\star} \bar{L}_{2k}^{(n-p)} \bar{\star}\right).$$

By Theorem 4.3.3, the latter identity is equivalent to

$$\left(-\frac{n}{2} + p + k\right) D_{2N}^{(p \rightarrow p-1)}\left(k - \frac{n}{2}\right) = D_{2N-2k}^{(p \rightarrow p-1)}\left(-k - \frac{n}{2}\right) \bar{\star}^2 \left(\bar{\star} \bar{L}_{2k}^{(n-p)} \bar{\star}\right).$$

Combining this result with $\bar{\star} \bar{L}_{2k}^{(n-p)} \bar{\star} = -\bar{\star}^2 \bar{L}_{2k}^{(p)}$ proves (6.10). Similarly, the first identity in (6.9) yields

$$\star D_{2N}^{(n-p \rightarrow n-p)}\left(N - \frac{n-1}{2}\right) \bar{\star} = -\star L_{2N}^{(n-p)} \iota^* \bar{\star}.$$

By Theorem 4.3.3, this identity is equivalent to

$$(-1)^{np} D_{2N}^{(p \rightarrow p-1)}\left(N - \frac{n-1}{2}\right) = -\left(\star L_{2N}^{(n-p)} \star\right) \star^2 (\star \iota^* \bar{\star}).$$

Now, using $\star L_{2N}^{(n-p)} \star = -\star^2 L_{2N}^{(p-1)}$ and Lemma 4.3.2, we find

$$D_{2N}^{(p \rightarrow p-1)}\left(N - \frac{n-1}{2}\right) = -L_{2N}^{(p-1)} \iota^* i_{\partial_n}.$$

This proves the first identity in (6.12).

We omit the analogous proofs of (6.11) and of the second identity in (6.12). \square

The main factorizations for even-order families of the first and the second type have analogs for odd-order families.

Theorem 6.2.3. *Let $N \in \mathbb{N}$ and $p = 0, \dots, n-1$. Then the odd-order families of the first type satisfy the factorization identities*

$$\left(\frac{n}{2} - p + k\right) D_{2N+1}^{(p \rightarrow p)}\left(k - \frac{n}{2}\right) = D_{2N+1-2k}^{(p \rightarrow p)}\left(-k - \frac{n}{2}\right) \circ \bar{L}_{2k}^{(p)} \quad (6.15)$$

and

$$\left(\frac{n-1}{2} - p - k\right) D_{2N+1}^{(p \rightarrow p)}\left(2N+1-k-\frac{n-1}{2}\right) = L_{2k}^{(p)} \circ D_{2N+1-2k}^{(p \rightarrow p)}\left(2N+1-k-\frac{n-1}{2}\right) \quad (6.16)$$

for $k = 1, \dots, N$.

Proof. We apply Theorem 5.3.1. Similar arguments as in the proof of Theorem 6.2.1 show that for the proof of (6.15) it suffices to prove

$$\begin{aligned} \beta_i^{(N)}\left(k - \frac{n}{2}\right) &= 0, \quad i = 0, \dots, k-1, \\ \beta_i^{(N)}\left(k - \frac{n}{2}\right) &= \beta_{i-k}^{(N-k)}\left(-k - \frac{n}{2}\right), \quad i = k, \dots, N \end{aligned}$$

and

$$\begin{aligned} \gamma_i^{(N)}\left(k - \frac{n}{2}\right) &= 0, \quad i = 0, \dots, k-1, \\ \gamma_i^{(N)}\left(k - \frac{n}{2}\right) &= \gamma_{i-k}^{(N-k)}\left(-k - \frac{n}{2}\right), \quad i = k+1, \dots, N. \end{aligned}$$

But the first set of assertions follows directly from the definition (5.6). In turn, the second set of assertions follows by combining the first set with Remark 5.3.2. Similarly, for the proof of (6.16) it suffices to prove

$$\beta_i^{(N)}\left(2N+1-k-\frac{n-1}{2}\right) = 0, \quad i = N-k+1, \dots, N,$$

$$\beta_i^{(N)}(2N+1-k-\frac{n-1}{2}) = \beta_i^{(N-k)}(2N+1-k-\frac{n-1}{2}), \quad i = 0, \dots, N-k$$

and

$$\begin{aligned} \gamma_i^{(N)}(2N+1-k-\frac{n-1}{2}) &= 0, \quad i = N-k+1, \dots, N, \\ \gamma_i^{(N)}(2N+1-k-\frac{n-1}{2}) &= \gamma_i^{(N-k)}(2N+1-k-\frac{n-1}{2}), \quad i = 0, \dots, N-k. \end{aligned}$$

Again, the first set of relations directly follows from the definition (5.6) and the second set follows by combining the first set with Remark 5.3.2. \square

Now Hodge conjugation yields

Theorem 6.2.4. *Let $N \in \mathbb{N}_0$ and $p = 1, \dots, n$. Then the odd-order families of the second type satisfy the factorization identities*

$$(\frac{n}{2}-p-k)D_{2N+1}^{(p \rightarrow p-1)}(k-\frac{n}{2}) = D_{2N+1-2k}^{(p \rightarrow p-1)}(-k-\frac{n}{2}) \circ \bar{L}_{2k}^{(p)} \quad (6.17)$$

and

$$(\frac{n+1}{2}-p+k)D_{2N+1}^{(p \rightarrow p-1)}(2N+1-k-\frac{n-1}{2}) = L_{2k}^{(p-1)} \circ D_{2N+1-2k}^{(p \rightarrow p-1)}(2N+1-k-\frac{n-1}{2}) \quad (6.18)$$

for $k = 1, \dots, N$.

Finally, we reformulate Theorem 6.2.1 and Theorem 6.2.2 so that the numerical coefficients on the left-hand sides of the factorization identities disappear. For the families of the first type we find the following result.

Theorem 6.2.5. *Assume that n is even and $p < \frac{n}{2}$. Then the identities in Theorem 6.2.1 are equivalent to the factorizations*

$$\tilde{D}_{2N}^{(p \rightarrow p)}(k-\frac{n}{2}) = \tilde{D}_{2N-2k}^{(p \rightarrow p)}(-k-\frac{n}{2}) \circ \tilde{\tilde{L}}_{2k}^{(p)}, \quad (6.19)$$

$$\tilde{D}_{2N}^{(p \rightarrow p)}(2N-k-\frac{n-1}{2}) = \tilde{\tilde{L}}_{2k}^{(p)} \circ \tilde{D}_{2N-2k}^{(p \rightarrow p)}(2N-k-\frac{n-1}{2}) \quad (6.20)$$

for $k = 1, \dots, N$ with the renormalized families

$$\tilde{D}_{2N}^{(p \rightarrow p)}(\lambda) \stackrel{\text{def}}{=} \frac{D_{2N}^{(p \rightarrow p)}(\lambda)}{\lambda + p - 2N}$$

and the renormalized Branson-Gover operators

$$\tilde{\tilde{L}}_{2N}^{(p)} \stackrel{\text{def}}{=} \frac{L_{2N}^{(p)}}{\frac{n}{2}-p+N} = (\delta d)^N + \dots \quad (6.21)$$

on a manifold of dimension n .

We stress that (6.19) and (6.20) also include the case $k = N$ with $\tilde{D}_0^{(p \rightarrow p)}(\lambda) = \iota^*$. The assumptions in Theorem 6.2.5 guarantee that both sides of the factorization identities are well-defined. Theorem 6.2.5 resembles the factorization identities for residue families on functions [J09], [J13].

For the second type families we have the following analogous result.

Theorem 6.2.6. *Assume that n is even and $p < \frac{n}{2}$. Then the identities in Theorem 6.2.2 are equivalent to the factorizations*

$$\tilde{D}_{2N}^{(p \rightarrow p-1)}(k-\frac{n}{2}) = \tilde{D}_{2N-2k}^{(p \rightarrow p-1)}(-k-\frac{n}{2}) \circ \tilde{\tilde{L}}_{2k}^{(p)}, \quad (6.22)$$

$$\tilde{D}_{2N}^{(p \rightarrow p-1)}(2N-k-\frac{n-1}{2}) = \tilde{\tilde{L}}_{2k}^{(p-1)} \circ \tilde{D}_{2N-2k}^{(p \rightarrow p-1)}(2N-k-\frac{n-1}{2}) \quad (6.23)$$

for $k = 1, \dots, N$ with the renormalized families

$$\tilde{D}_{2N}^{(p \rightarrow p-1)}(\lambda) \stackrel{\text{def}}{=} \frac{D_{2N}^{(p \rightarrow p-1)}(\lambda)}{\lambda + n - p}.$$

Similarly as above, the identities (6.22) and (6.23) include the case $k = N$ with $\tilde{D}_0^{(p \rightarrow p-1)}(\lambda) = -\iota^* i_{\partial_n}$. The assumptions in Theorem 6.2.6 guarantee that both sides of the factorization identities are well-defined.

We omit the formulation of the odd-order analogs of these results.

6.3. Supplementary factorizations. In the present section, we establish additional factorization identities for the conformal symmetry breaking families. These involve the four geometric operators d , δ , \bar{d} and $\bar{\delta}$ as factors.

We first formulate the identities which involve even-order families of the first type, i.e., families of the form $D_{2N}^{(p \rightarrow p)}(\lambda)$.

Theorem 6.3.1. *For $N \in \mathbb{N}$, we have the factorization identities*

$$D_{2N}^{(p \rightarrow p)}(-p+2N) = -(2N)dD_{2N-1}^{(p \rightarrow p-1)}(-p+2N), \quad 1 \leq p \leq n-1, \quad (6.24)$$

$$D_{2N}^{(p \rightarrow p)}(-p) = (2N)D_{2N-1}^{(p+1 \rightarrow p)}(-p-1)\bar{d}, \quad 0 \leq p \leq n-1. \quad (6.25)$$

Moreover, for $N \in \mathbb{N}_0$, we have

$$(n-2p-2N-1)D_{2N+1}^{(p \rightarrow p-1)}(p-n+2N+1) = \delta D_{2N}^{(p \rightarrow p)}(p-n+2N+1) \quad (6.26)$$

for $1 \leq p \leq n-1$ and

$$(n-2p+2N)D_{2N+1}^{(p+1 \rightarrow p)}(-n+p+1) = D_{2N}^{(p \rightarrow p)}(-n+p)\bar{\delta} \quad (6.27)$$

for $0 \leq p \leq n-1$.

We stress that all factors on the right-hand sides of these identities are conformally equivariant. For instance, the individual factors on the right-hand side of (6.24) intertwine

$$D_{2N-1}^{(p \rightarrow p-1)}(-p+2N) : d\pi_{-2N}^{(p)} \rightarrow d\pi_0'^{(p-1)} \quad \text{and} \quad d : d\pi_0'^{(p-1)} \rightarrow d\pi_0'^{(p)}.$$

Similarly, the factors on the right-hand side of (6.27) intertwine

$$D_{2N}^{(p \rightarrow p)}(-n+p) : d\pi_{n-2p}^{(p)} \rightarrow d\pi_{n-2p+2N}'^{(p)} \quad \text{and} \quad \bar{\delta} : d\pi_{n-2p-2}^{(p+1)} \rightarrow d\pi_{n-2p}^{(p)}.$$

Note that the latter intertwining relation is a consequence of general properties of the co-differential. In fact, on (M^n, g) we have $\delta_{\gamma_*g} = \gamma_*\delta_g\gamma^*$ for any diffeomorphisms γ and

$$\delta_{e^{2\varphi}g} \circ e^{-(n-2p)\varphi} = e^{-(n-2p+2)\varphi} \circ \delta_g \quad \text{on } \Omega^p(M)$$

for any conformal change of g .

Note that the relations (6.25) are equivalent to the commutative triangles mentioned in [J01, page 593].

The following proof of Theorem 6.3.1 shows that all results are direct consequences of the geometric formulas in Section 5.

Proof. The assertions follow from Theorem 5.2.1 and Theorem 5.3.3. We start with the proof of (6.24). On the one hand, Theorem 5.3.3 implies

$$dD_{2N-1}^{(p \rightarrow p-1)}(-p+2N) = - \sum_{i=1}^{N-1} \gamma_i^{(N-1)}(-p+2N; n-p)(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i$$

$$\begin{aligned}
& - (n-2p+2N) \sum_{i=1}^{N-1} \beta_i^{(N-1)}(-p+2N)(d\delta)^{N-i} \iota^*(\bar{\delta}\bar{d})^i \\
& + (n-2p+1) \beta_{N-1}^{(N-1)}(-p+2N) \iota^*(\bar{d}\bar{\delta})^N \\
& - (n-2p+2N) \beta_0^{(N-1)}(-p+2N)(d\delta)^N \iota^*.
\end{aligned} \tag{6.28}$$

On the other hand, Theorem 5.2.1 gives

$$\begin{aligned}
D_{2N}^{(p \rightarrow p)}(-p+2N) &= \sum_{i=1}^{N-1} (2N-2i) \alpha_i^{(N)}(-p+2N)(d\delta)^{N-i} \iota^*(\bar{\delta}\bar{d})^i \\
&+ 2N \sum_{i=1}^{N-1} \alpha_i^{(N)}(-p+2N)(d\delta)^{N-i} \iota^*(\bar{d}\bar{\delta})^i \\
&+ 2N \alpha_0^{(N)}(-p+2N)(d\delta)^N \iota^* \\
&+ 2N \alpha_N^{(N)}(-p+2N) \iota^*(\bar{d}\bar{\delta})^N.
\end{aligned} \tag{6.29}$$

Now we have the relations

$$\gamma_i^{(N-1)}(-p+2N; n-p) = \alpha_i^{(N)}(-p+2N), \quad i = 1, \dots, N-1 \tag{6.30}$$

and

$$2N(n-2p+2N) \beta_i^{(N-1)}(-p+2N) = (2N-2i) \alpha_i^{(N)}(-p+2N), \quad i = 0, \dots, N-1. \tag{6.31}$$

Indeed, (5.17) implies

$$\begin{aligned}
\gamma_i^{(N-1)}(-p+2N; n-p) &= (-1)^i 2^{N-1} \frac{(N-1)!}{(2N-1)!} \binom{N}{i} \\
&\times (n+2N-2p)(n-2p+2N-2i+1) \prod_{k=i+1}^{N-1} (-2p+4N+n-2k) \prod_{k=1}^{i-1} (n-2p+2N-2k+1).
\end{aligned}$$

But the last line can be simplified by extending the products up to $k = N$ and $k = i$, respectively. The result coincides with $\alpha_i^{(N)}(-p+2N)$. This proves (6.30). Next, the left-hand side of (6.31) equals

$$\begin{aligned}
& 2N(-1)^i 2^{N-1} \frac{(N-1)!}{(2N-1)!} \binom{N-1}{i} \\
& \times (n+2N-2p) \prod_{k=i+1}^{N-1} (-2p+n+4N-2k) \prod_{k=1}^i (-2p+2N+n-2k+1).
\end{aligned}$$

The last line can be simplified by extending the first product up to $k = N$. By simplification, the result coincides with the right-hand side of (6.31).

In addition, we have the relation

$$\alpha_N^{(N)}(-p+2N) = -(n-2p+1) \beta_{N-1}^{(N-1)}(-p+2N). \tag{6.32}$$

In fact, by definition we have

$$-(n-2p+1) \beta_{N-1}^{(N-1)}(-p+2N) = (-1)^N 2^{N-1} \frac{(N-1)!}{(2N-1)!} (n-2p+1) \prod_{k=1}^{N-1} (n-2p+2N-2k+1).$$

The latter formula simplifies by extending the product up to $k = N$. The result coincides with $\alpha_N^{(N)}(-p + N)$.

Now (6.24) follows by combining (6.28) and (6.29) with the relations (6.30)–(6.32).

Next, we prove (6.25). Theorem 5.3.3 implies

$$\begin{aligned}
D_{2N-1}^{(p+1 \rightarrow p)}(-p-1)\bar{d} &= - \sum_{i=1}^{N-1} \gamma_i^{(N-1)}(-p-1; n-p-1)(\delta d)^{N-i} \iota^*(\bar{\delta}\bar{d})^i \\
&\quad + (n-2p-2N-1) \sum_{i=0}^{N-2} \beta_i^{(N-1)}(-p-1)(\delta d)^{N-1-i} \iota^*(\bar{\delta}\bar{d})^{i+1} \\
&\quad + (n-2p-2N-1) \beta_{N-1}^{(N-1)}(-p-1) \iota^*(\bar{\delta}\bar{d})^N \\
&\quad - (n-2p-2) \beta_0^{(N-1)}(-p-1)(\delta d)^N \iota^*.
\end{aligned} \tag{6.33}$$

But Theorem 5.2.1 yields

$$\begin{aligned}
D_{2N}^{(p \rightarrow p)}(-p) &= -2N \sum_{i=1}^{N-1} \alpha_i^{(N)}(-p)(\delta d)^{N-i} \iota^*(\bar{\delta}\bar{d})^i \\
&\quad - \sum_{i=1}^{N-1} 2i \alpha_i^{(N)}(-p)(\delta d)^{N-i} \iota^*(\bar{\delta}\bar{d})^i \\
&\quad - 2N \alpha_N^{(N)}(-p) \iota^*(\bar{\delta}\bar{d})^N \\
&\quad - 2N \alpha_0^{(N)}(-p)(\delta d)^N \iota^*.
\end{aligned} \tag{6.34}$$

By combining (6.33) and (6.34) with the relations

$$\gamma_i^{(N-1)}(-p-1; n-p-1) = \alpha_i^{(N)}(-p), \quad i = 1, \dots, N-1, \tag{6.35}$$

$$2N(n-2p-2N-1) \beta_{i-1}^{(N-1)}(-p-1) = -2i \alpha_i^{(N)}(-p), \quad i = 1, \dots, N \tag{6.36}$$

and

$$(n-2p-2) \beta_0^{(N-1)}(-p-1) = \alpha_0^{(N)}(-p), \tag{6.37}$$

we find

$$D_{2N}^{(p \rightarrow p)}(-p) = 2N D_{2N-1}^{(p+1 \rightarrow p)}(-p-1)\bar{d}.$$

This proves (6.25).

Next, we prove the relations (6.35)–(6.37). By (5.17), we have

$$\begin{aligned}
\gamma_i^{(N-1)}(-p-1; n-p-1) &= (-1)^i 2^{N-1} \frac{(N-1)!}{(2N-1)!} \binom{N}{i} \\
&\quad \times (n-2p-2N-1)(n-2p-2i-2) \prod_{k=i+1}^{N-1} (n-2p-2k-2) \prod_{k=1}^{i-1} (n-2p-2k-2N-1).
\end{aligned}$$

The last line simplifies by letting the products run from $k = i$ and $k = 0$, respectively. The result coincides with $-\alpha_i^{(N)}(-p)$. This proves (6.35). Next, we have

$$2N(n-2p-2N-1) \beta_{i-1}^{(N-1)}(-p-1) = (-1)^{i-1} 2^N \frac{N!}{(2N-1)!} \binom{N-1}{i-1}$$

$$\times (n-2p-2N-1) \prod_{k=i}^{N-1} (n-2p-2k-2) \prod_{k=1}^{i-1} (n-2p-2N-2k-1).$$

Again, the last line simplifies by letting the last product run from $k = 0$. Further simplification confirms (6.36). Finally, (6.37) is easy to verify.

We continue with the proof of (6.26). On the one hand, Theorem 5.3.3 implies

$$\begin{aligned} D_{2N+1}^{(p \rightarrow p-1)}(p-n+2N+1) &= - \sum_{i=1}^N \gamma_i^{(N)}(p-n+2N+1; n-p) (\delta d)^{N-i} \delta \iota^* (\bar{d} \bar{\delta})^i \\ &\quad - (2N+1) \sum_{i=1}^{N-1} \beta_i^{(N)}(p-n+2N+1) (\delta d)^{N-i} \delta \iota^* (\bar{d} \bar{\delta})^i \\ &\quad - (2N+1) \beta_0^{(N)}(p-n+2N+1) (\delta d)^N \delta \iota^* \\ &\quad - (2N+1) \beta_N^{(N)}(p-n+2N+1) \iota^* (\bar{d} \bar{\delta})^N. \end{aligned} \quad (6.38)$$

On the other hand, Theorem 5.2.1 yields

$$\begin{aligned} \delta D_{2N}^{(p \rightarrow p)}(p-n+2N+1) &= (2p-n+2N+1) \sum_{i=1}^N \alpha_i^{(N)}(p-n+2N+1) \delta (d \delta)^{N-i} \iota^* (\bar{d} \bar{\delta})^i \\ &\quad + (2p-n+2N+1) \alpha_0^{(N)}(p-n+2N+1) \delta (d \delta)^N \iota^* \\ &\quad + (2p-n+1) \alpha_N^{(N)}(p-n+2N+1) \delta \iota^* (\bar{d} \bar{\delta})^N \\ &\quad + \sum_{i=1}^{N-1} (2p-n+2N+1-2i) \alpha_i^{(N)}(p-n+2N+1) (\delta d)^{N-i} \delta \iota^* (\bar{d} \bar{\delta})^i. \end{aligned} \quad (6.39)$$

But we have the relations

$$\gamma_i^{(N)}(p-n+2N+1; n-p) = \alpha_i^{(N)}(p-n+2N+1) \quad (6.40)$$

for $i = 1, \dots, N$ and

$$-(n-2p-2N-1)(2N+1) \beta_i^{(N)}(p-n+2N+1) = (2p-n+2N-2i+1) \alpha_i^{(N)}(p-n+2N+1) \quad (6.41)$$

for $i = 0, \dots, N$.

The assertion (6.26) follows by combining (6.38) and (6.39) with (6.40) and (6.41).

Now, we prove the relations (6.40) and (6.41). The definition (5.17) yields

$$\begin{aligned} \gamma_i^{(N)}(p-n+2N+1; n-p) &= (-1)^i 2^N \frac{N!}{(N+1)(2N)!} \binom{N+1}{i} (N-i+1) \\ &\quad \times (2p-n+2N+1) \prod_{k=i+1}^N (2p-n+4N-2k+2) \prod_{k=1}^{i-1} (2p-n+2N-2k+1). \end{aligned}$$

The second line simplifies by letting the last product run from $k = 0$. Further simplification confirms (6.40). Next, we find

$$(2p-n+2N-2i+1) \alpha_i^{(N)}(p-n+2N+1) = (-1)^i 2^N \frac{N!}{(2N)!} \binom{N}{i}$$

$$\times (2p-n+2N-2i+1) \prod_{k=i+1}^N (2p-n+4N-2k+2) \prod_{k=1}^i (2p-n+2N-2k+3)$$

and

$$\begin{aligned} (n-2p-2N-1)(2N+1)\beta_i^{(N)}(p-n+2N+1) &= (-1)^i 2^N \frac{N!}{(2N)!} \binom{N}{i} \\ &\times (n-2p-2N-1) \prod_{k=i+1}^N (2p-n+4N-2k+2) \prod_{k=1}^i (2p-n+2N-2k+1). \end{aligned}$$

These formulas easily imply (6.41).

Finally, we prove (6.27). On the one hand, Theorem 5.3.3 implies

$$\begin{aligned} D_{2N+1}^{(p+1 \rightarrow p)}(-n+p+1) &= - \sum_{i=1}^N \gamma_i^{(N)}(-n+p+1; n-p-1) (\delta d)^{N-i} \delta \iota^* (\bar{d} \bar{\delta})^i \\ &\quad - (2N+1) \sum_{i=1}^{N-1} \beta_i^{(N)}(-n+p+1) (d\delta)^{N-i} \iota^* \bar{\delta} (\bar{d} \bar{\delta})^i \\ &\quad - (2N+1) \beta_0^{(N)}(-n+p+1) (d\delta)^N \iota^* \bar{\delta} \\ &\quad - (2N+1) \beta_N^{(N)}(-n+p+1) \iota^* \bar{\delta} (\bar{d} \bar{\delta})^N. \end{aligned} \quad (6.42)$$

On the other hand, Theorem 5.2.1 gives

$$\begin{aligned} D_{2N}^{(p \rightarrow p)}(-n+p) \bar{\delta} &= \sum_{i=1}^{N-1} (-n+2p-2i) \alpha_i^{(N)}(-n+p) (d\delta)^{N-i} \iota^* (\bar{\delta} \bar{d})^i \bar{\delta} \\ &\quad + (-n+2p-2N) \sum_{i=1}^N \alpha_{i-1}^{(N)}(-n+p) (\delta d)^{N-i} \delta \iota^* (\bar{d} \bar{\delta})^i \\ &\quad + (-n+2p) \alpha_0^{(N)}(-n+p) (d\delta)^N \iota^* \bar{\delta} \\ &\quad + (-n+2p-2N) \alpha_N^{(N)}(-n+p) \iota^* (\bar{\delta} \bar{d})^N \bar{\delta} \end{aligned} \quad (6.43)$$

Combining (6.42) and (6.43) with the relations

$$\gamma_i^{(N)}(-n+p+1; n-p-1) = \alpha_{i-1}^{(N)}(-n+p), \quad i = 1, \dots, N \quad (6.44)$$

and

$$(n-2p+2N)(2N+1)\beta_i^{(N)}(-n+p+1) = (n-2p+2i)\alpha_i^{(N)}(-n+p), \quad i = 0, \dots, N \quad (6.45)$$

completes the proof of (6.27).

Now we prove (6.44) and (6.45). Equation (5.17) implies that

$$\begin{aligned} \gamma_i^{(N)}(-n+p+1; n-p-1) &= (-1)^i 2^N \frac{N!}{(2N+1)!} \frac{1}{N+1} \binom{N+1}{i} i(2N+1) \\ &\quad \times (n-2p+2N) \prod_{k=i+1}^N (-n+2p-2k+2) \prod_{k=1}^{i-1} (-n+2p-2k-2N+1). \end{aligned}$$

The second line simplifies by letting the first product run up to $k = N + 1$. Further simplification shows that the result coincides with $\alpha_{i-1}^{(N)}(-n+p)$. This proves (6.44). Finally, the left-hand side of (6.45) equals

$$(-1)^i 2^N \frac{N!}{(2N)!} \binom{N}{i} (n-2p+2N) \prod_{k=i+1}^N (-n+2p-2k+2) \prod_{k=1}^i (-n+2p-2k-2N+1).$$

The last expression simplifies by letting the first product run up to $k = N + 1$. On the other hand, the right-hand side of (6.45) equals

$$(-1)^i 2^N \frac{N!}{(2N)!} \binom{N}{i} (n-2p+2i) \prod_{k=i+1}^N (-n+2p-2k) \prod_{k=1}^i (-n+2p-2N-2k+1).$$

The latter expression simplifies by letting the first product run from $k = i$. This proves (6.45). The proof is complete. \square

By Hodge conjugation, Theorem 6.3.1 implies the following factorization identities which involve even-order families of the second type.

Theorem 6.3.2. *For $N \in \mathbb{N}$, we have the factorization identities*

$$\begin{aligned} D_{2N}^{(p \rightarrow p-1)}(p-n+2N) &= -(2N) \delta D_{2N-1}^{(p \rightarrow p)}(p-n+2N), \quad 1 \leq p \leq n-1, \\ D_{2N}^{(p \rightarrow p-1)}(p-n) &= (2N) D_{2N-1}^{(p-1 \rightarrow p-1)}(p-n) \bar{d}, \quad 1 \leq p \leq n. \end{aligned}$$

Moreover, for $N \in \mathbb{N}_0$, we have

$$\begin{aligned} (-n+2p-2N-1) D_{2N+1}^{(p \rightarrow p)}(-p+2N+1) &= d D_{2N}^{(p \rightarrow p-1)}(-p+2N+1), \quad 1 \leq p \leq n-1, \\ (n-2p-2N-2) D_{2N+1}^{(p \rightarrow p)}(-p) &= D_{2N}^{(p+1 \rightarrow p)}(-p-1) \bar{d}, \quad 0 \leq p \leq n-1. \end{aligned}$$

Proof. The results follow by combining Theorem 4.3.3 with Theorem 6.3.1. We omit the details. \square

Remark 6.3.3. *Theorem 6.3.1 and Theorem 6.3.2 have extensions to general metrics in terms of residue families [FJS16]. From that point of view, the following arguments are of interest. The left-hand side of (6.26) vanishes for odd n and $p = \frac{n-1}{2} - N$. Hence*

$$\delta D_{2N}^{(p \rightarrow p)}(-\frac{n-1}{2} + N) = 0.$$

By Theorem 6.2.1, it follows that

$$\delta L_{2N}^{(\frac{n-1}{2}-N)} \iota^* = 0.$$

But this vanishing follows from the double factorization (6.2) of the critical Branson-Gover operators on a manifold of even dimension n . Similarly, for even n and $p = \frac{n}{2} - N - 1$, the last identity in Theorem 6.3.2 implies

$$D_{2N}^{(\frac{n}{2}-N \rightarrow \frac{n}{2}-N-1)}(-\frac{n}{2} + N) \bar{d} = 0.$$

By Theorem 6.2.2, it follows that

$$\iota^* i_{\partial_n} \bar{L}_{2N}^{(\frac{n}{2}-N)} \bar{d} = 0.$$

Again this vanishing result should be regarded as a consequence of the double factorization of the critical Branson-Gover operators.

6.4. Applications. In the present section, we describe two applications to gauge companion and Q -curvature operators.

Equation (6.9) in Theorem 6.2.1 and Equation (6.12) in Theorem 6.2.2 show that even-order symmetry breaking operators of both types specialize at certain arguments to Branson-Gover operators. The following result provides an analogous description of the gauge companion operators in terms of odd-order conformal symmetry breaking operators of the second type.

Theorem 6.4.1. *Assume that $n - 1$ is even and that $n - 2p \geq 1$. Then the restriction of $D_{n-2p}^{(p \rightarrow p-1)}(-p)$ to the space of closed p -forms is given by the gauge companion operator $G_{n-2p}^{(p)}$. More precisely, we have*

$$D_{n-2p}^{(p \rightarrow p-1)}(-p)|_{\ker(\bar{d})} = -G_{n-2p}^{(p)}\iota^*. \quad (6.46)$$

Proof. Let $n - 2p = 2N + 1$. Theorem 5.3.3 implies

$$D_{n-2p}^{(p \rightarrow p-1)}(-p)|_{\ker(\bar{d})} = -\sum_{i=1}^N \gamma_i^{(N)}(-p; n-p)(\delta d)^{N-i} \delta \iota^* (\bar{d}\bar{\delta})^i - (n-2p)\beta_0^{(N)}(-p)(\delta d)^N \delta \iota^*.$$

But

$$\gamma_i^{(N)}(-p; n-p) = 0 \quad \text{for } i \geq 1 \quad \text{and} \quad (n-2p)\beta_0^{(N)}(-p) = 1$$

yield the result

$$D_{n-2p}^{(p \rightarrow p-1)}(-p)|_{\ker(\bar{d})} = -\delta(d\delta)^{\frac{n-1}{2}-p}\iota^*.$$

This proves the assertion. \square

Example 6.4.2. *We illustrate Theorem 6.4.1 in low-order special cases. Using Example 4.8.1, we get*

$$\begin{aligned} D_1^{(p \rightarrow p-1)}(-p) &= -\delta \iota^* = -G_1^{(p)}\iota^* \quad \text{for } n - 2p = 1, \\ D_3^{(p \rightarrow p-1)}(-p)|_{\ker(\bar{d})} &= -\delta d \delta \iota^* = -G_3^{(p)}\iota^* \quad \text{for } n - 2p = 3. \end{aligned}$$

Note that for the first-order operator the restriction to the kernel of \bar{d} is unnecessary.

We continue with a discussion of Q -curvature operators. We observe that the supplementary factorization (6.25) (or Theorem 5.2.1) shows that the restriction of $D_{2N}^{(p \rightarrow p)}(\lambda)$ to $\ker(\bar{d})$ vanishes at $\lambda = -p$. This motivates the following definition.

Definition 6.4.3. *The Q -curvature polynomial*

$$Q_{2N}^{(p)}(\lambda) : \Omega^p(\mathbb{R}^n)|_{\ker(\bar{d})} \rightarrow \Omega^p(\mathbb{R}^{n-1})$$

is defined by

$$(\lambda + p)Q_{2N}^{(p)}(\lambda) \stackrel{\text{def}}{=} D_{2N}^{(p \rightarrow p)}(\lambda)|_{\ker(\bar{d})}. \quad (6.47)$$

Definition 6.4.3 extends the notion of Q -curvature polynomials introduced in [J09]. We recall that, for general metrics, we define $Q_{2N}^{res}(\lambda) \stackrel{\text{def}}{=} D_{2N}^{res}(\lambda)(1)$. The polynomial $Q_{2N}^{res}(\lambda)$ vanishes at $\lambda = 0$. For the Euclidean metric g_0 , however, all Q -curvature polynomials $Q_{2N}^{res}(\lambda)$ vanish identically. The fact that this is no longer the case for $p > 0$ follows from the following consequence of Theorem 5.2.1.

Corollary 6.4.4. *The polynomials $Q_{2N}^{(p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1})$ are given by the explicit formula*

$$Q_{2N}^{(p)}(\lambda) = \sum_{i=0}^N \alpha_i^{(N)}(\lambda) (d\delta)^{N-i} \iota^* (\bar{d}\bar{\delta})^i$$

with the coefficients $\alpha_i^{(N)}$ as defined by (5.5).

Example 6.4.5. *We have*

$$Q_2^{(p)}(\lambda) = (2\lambda+n-2)d\delta\iota^* - (2\lambda+n-3)\iota^*\bar{d}\bar{\delta}$$

and

$$Q_4^{(p)}(\lambda) = \frac{1}{3} [(2\lambda+n-2)(2\lambda+n-4)(d\delta)^2\iota^* - 2(2\lambda+n-4)(2\lambda+n-5)(d\delta)\iota^*(\bar{d}\bar{\delta}) + (2\lambda+n-5)(2\lambda+n-7)(\bar{d}\bar{\delta})^2].$$

Corollary 6.4.4 shows that, for general λ , the Q -curvature polynomial $Q_{2N}^{(p)}(\lambda)$ is *not* a tangential operator with respect to the subspace \mathbb{R}^{n-1} . However, for certain values of λ it reduces to a tangential operator. In fact, we have the following result.

Corollary 6.4.6. *At the argument $\lambda = N - \frac{n-1}{2}$, the Q -curvature polynomial $Q_{2N}^{(p)}(\lambda)$ reduces to a tangential operator. More precisely, we have the formula*

$$Q_{2N}^{(p)}(N - \frac{n-1}{2}) = (d\delta)^N \iota^*.$$

Proof. The result follows from Corollary 6.4.4 using

$$\alpha_i^{(N)}(N - \frac{n-1}{2}) = 0 \quad \text{for } i \geq 1$$

and

$$\alpha_0^{(N)}(N - \frac{n-1}{2}) = 1.$$

The proof is complete. \square

Corollary 6.4.6 shows that, for the Euclidean metric, the following definition of the Q -curvature operator using conformal symmetry breaking operators coincides with the definition by

$$L_{2N}^{(p)}|_{\ker d} = (\frac{n}{2} - N - p)Q_{2N}^{(p)}, \quad (6.48)$$

which generalizes the classical definition

$$P_{2N}(1) = (\frac{n}{2} - N)Q_{2N}$$

of the Q -curvature Q_{2N} . Note that (6.48) defines $Q_{2N}^{(p)}$ only if $(\frac{n}{2} - N - p) \neq 0$, i.e., only in the non-critical case.

Definition 6.4.7.

$$Q_{2N}^{(p)} \stackrel{\text{def}}{=} Q_{2N}^{(p)}(N - \frac{n-1}{2}) : \Omega^p(\mathbb{R}^{n-1})|_{\ker(d)} \rightarrow \Omega^p(\mathbb{R}^{n-1}). \quad (6.49)$$

The above observations also lead to the following result which should be considered as an analog of the holographic description of the critical Branson curvature Q_n of (M^n, g) in terms of $\dot{D}_n^{\text{res}}(0)(1)$ ([J09], [GJ07]); here dot denotes the derivative with respect to λ .

Theorem 6.4.8. *Assume that $n - 1$ is even and that $n - 2p \geq 3$. Then*

$$\dot{D}_{n-1-2p}^{(p \rightarrow p)}(-p)|_{\ker(\bar{d})} = Q_{n-1-2p}^{(p)} \iota^* \quad (6.50)$$

as an identity of operators on closed forms on \mathbb{R}^n .

Proof. On the one hand, (6.47) implies

$$\dot{D}_{2N}^{(p \rightarrow p)}(-p)|_{\ker(\bar{d})} = Q_{2N}^{(p)}(-p).$$

On the other hand, we have

$$Q_{2N}^{(p)}(N - \frac{n-1}{2}) = (d\delta)^N \iota^*$$

by Corollary 6.4.6. In the critical case $2N = n - 1 - 2p$, a comparison of both facts proves the assertion. \square

Finally, we give a proof of the double factorization property of the critical Branson-Gover operators for the Euclidean metric on \mathbb{R}^{n-1} from the perspective of conformal symmetry breaking operators.

Corollary 6.4.9. *Let $n - 1$ be even and $p < \frac{n-1}{2}$. Then the critical Branson-Gover operators $L_{n-2p-1}^{(p)}$ of the Euclidean metric on \mathbb{R}^{n-1} satisfy the double factorization identity*

$$L_{n-2p-1}^{(p)} = (n-2p-1) \delta Q_{n-2p-3}^{(p+1)} d,$$

where $Q_{n-2p-3}^{(p+1)} = (d\delta)^{\frac{n-3}{2}-p}$.

Proof. The supplementary factorizations (6.25) and (6.26) read

$$D_{2N}^{(p \rightarrow p)}(-p) = 2N D_{2N-1}^{(p+1 \rightarrow p)}(-p-1) \bar{d}, \quad (6.51)$$

and

$$(2N+2p-n+1) D_{2N-1}^{(p+1 \rightarrow p)}(p-n+2N) = -\delta D_{2N-2}^{(p+1 \rightarrow p+1)}(p-n+2N), \quad (6.52)$$

respectively. By Theorem 6.4.6, we have

$$D_{2N-2}^{(p+1 \rightarrow p+1)}(p-n+2N)|_{\ker(\bar{d})} = (2N+2p-n+1) Q_{2N-2}^{(p+1)}(p-n+2N).$$

Now assume that $(2N+2p-n+1) \neq 0$. Then the restriction of (6.52) to the subspace of closed forms gives, after division by the common factor,

$$D_{2N-1}^{(p+1 \rightarrow p)}(p-n+2N)|_{\ker(\bar{d})} = -\delta Q_{2N-2}^{(p+1 \rightarrow p+1)}(p-n+2N). \quad (6.53)$$

Next, we apply analytic continuation in the dimension n and conclude that if $2N = n - 2p - 1$ then (6.51) and (6.53) combine into

$$D_{n-2p-1}^{(p \rightarrow p)}(-p) = -(n-2p-1) \delta Q_{n-2p-3}^{(p+1)}(-p-1) \bar{d}.$$

An application of Theorem 6.2.1 to the left-hand side and an application of Theorem 6.4.6 to the right-hand side turn the last equation into

$$L_{n-2p-1}^{(p)} = (n-1-2p) \delta Q_{n-2p-3}^{(p+1)} d$$

using $d\iota^* = \iota^* \bar{d}$. This completes the proof. \square

Analogous arguments using the supplementary factorizations (6.24) and (6.27) prove the double factorization identity

$$\iota^* \bar{L}_{2p-n}^{(p)} = (n-2p) d\dot{D}_{2p-n-2}^{(p-1 \rightarrow p-1)} (-n+p-1) \bar{\delta} \quad (6.54)$$

for even n and $\frac{n}{2} < p \leq n-1$. Alternatively, (6.54) is a direct consequence of Theorem 5.2.1.

APPENDIX: GEGENBAUER AND JACOBI POLYNOMIALS

We summarize basic conventions and properties concerning Gegenbauer and Jacobi polynomials.

First, we recall that the Pochhammer symbol of $a \in \mathbb{C}$ is defined by

$$(a)_l \stackrel{\text{def}}{=} a(a+1) \cdots (a+l-1)$$

for $l \in \mathbb{N}$, and $(a)_0 \stackrel{\text{def}}{=} 1$. Then the generalized hypergeometric function ${}_pF_q$ of type (p, q) , $p, q \in \mathbb{N}$, is defined by the Taylor series

$${}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right] \stackrel{\text{def}}{=} \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_p)_l}{(b_1)_l \cdots (b_q)_l} \frac{z^l}{l!}, \quad (6.55)$$

for $a_i \in \mathbb{C}$ ($1 \leq i \leq p$), $b_j \in \mathbb{C} \setminus \{-\mathbb{N}_0\}$ ($1 \leq j \leq q$), and $z \in \mathbb{C}$.

Let $m \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}$ such that $(\alpha+1)_m \neq 0$. The corresponding Jacobi polynomial of degree m is defined by

$$P_m^{(\alpha, \beta)}(z) \stackrel{\text{def}}{=} \frac{(\alpha+1)_m}{m!} {}_2F_1 \left[\begin{matrix} -m, 1+\alpha+\beta+m \\ \alpha+1 \end{matrix}; \frac{1-z}{2} \right]. \quad (6.56)$$

A specialization of Jacobi polynomials leads to Gegenbauer polynomials:

$$\begin{aligned} C_m^\alpha(z) &= \frac{(2\alpha)_m}{(\alpha + \frac{1}{2})_m} P_m^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(z) \\ &= \frac{(2\alpha)_m}{m!} {}_2F_1 \left[\begin{matrix} -m, 2\alpha+m \\ \alpha + \frac{1}{2} \end{matrix}; \frac{1-z}{2} \right]. \end{aligned} \quad (6.57)$$

An explicit formula for Gegenbauer polynomials (see [BE53, Section 3.15, formula (9)]) reads

$$C_m^\alpha(z) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{\Gamma(m-k+\alpha)}{\Gamma(\alpha)k!(m-2k)!} (2z)^{m-2k}. \quad (6.58)$$

In particular, we have the even polynomials

$$\frac{N!}{(-\lambda - \frac{n-1}{2})_N} C_{2N}^{-\lambda - \frac{n-1}{2}}(z) = \sum_{j=0}^N a_j^{(N)}(\lambda) (-1)^j z^{2N-2j}$$

and the odd polynomials

$$\frac{N!}{2(-\lambda - \frac{n-1}{2})_{N+1}} C_{2N+1}^{-\lambda - \frac{n-1}{2}}(z) = \sum_{j=0}^N b_j^{(N)}(\lambda) (-1)^j z^{2N+1-2j}$$

with the coefficients

$$a_j^{(N)}(\lambda) \stackrel{\text{def}}{=} (-2)^{N-j} \frac{N!}{j!(2N-2j)!} \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n + 1) a_N^{(N)}(\lambda) \quad (6.59)$$

and

$$b_j^{(N)}(\lambda) \stackrel{\text{def}}{=} (-2)^{N-j} \frac{N!}{j!(2N-2j+1)!} \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n - 1) b_N^{(N)}(\lambda) \quad (6.60)$$

for $0 \leq j \leq N-1$ and $a_N^{(N)}(\lambda) = b_N^{(N)}(\lambda) = 1$ (see [J09, Theorems 5.1.2, 5.1.4]).

In the present paper, we also refer to the coefficients (6.59) and (6.60) with *non-trivial normalizations* $a_j^{(N)}(\lambda)$ and $b_j^{(N)}(\lambda)$ as *even* and *odd Gegenbauer coefficients*, respectively. However, we usually reserve the notation $a_j^{(N)}(\lambda)$ and $b_j^{(N)}(\lambda)$ for the coefficients (6.59) and (6.60) with the trivial normalizations $a_N^{(N)}(\lambda) = 1$ and $b_N^{(N)}(\lambda) = 1$. Gegenbauer coefficients with non-trivial normalizations will be denoted by different letters, e.g., $p_j^{(N)}(\lambda)$.

Accordingly (and by abuse of language) we shall also talk about Gegenbauer polynomials with non-trivial normalizations. Additional parameters in Gegenbauer coefficients are always separated by a semicolon “;”. The Gegenbauer coefficients satisfy the recurrence relations

$$(N-j+1)(2N-2j+1)a_{j-1}^{(N)}(\lambda) + j(2\lambda+n-4N+2j-1)a_j^{(N)}(\lambda) = 0, \quad (6.61)$$

$$(N-j+1)(2N-2j+3)b_{j-1}^{(N)}(\lambda) + j(2\lambda+n-4N+2j-3)b_j^{(N)}(\lambda) = 0 \quad (6.62)$$

for all $1 \leq j \leq N$.

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